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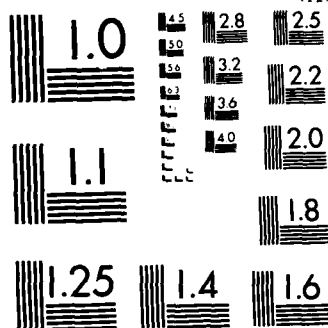
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COMPUTATIONAL TECHNIQUES FOR
THE ANALYSIS AND DESIGN OF
DIGITAL COMMUNICATION SYSTEMS

Final Technical Report

by

Ezio Biglieri

February 1985

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near receiving filter for transmission over a nonlinear channel. Finally, we consider trellis (Ungerboeck) codes. A new class of multidimensional codes is introduced, its application discussed, and an important property concerning their power spectral densities is shown.

ABSTRACT

In this report we introduce some computational techniques for the analysis and design of digital communication systems, possibly operating over a nonlinear channel (a typical example is a digital satellite system, with the transponder amplifier working at or near saturation for better efficiency). We first consider evaluation of error probabilities when the disturbances are only incompletely known. Then we describe a combined simulation-and-analytical technique to evaluate the power density spectrum of a digital signal passed through a nonlinear channel. Next, we consider the problem of designing and evaluating the performance of the optimum linear receiving filter for transmission over a nonlinear channel. Finally, we consider Ungerboeck codes: a new class of multidimensional codes is introduced, its applications discussed, and an important property concerning their power spectral densities is shown.

KEY WORDS: Digital communication systems; -- Nonlinear channels -- Channels with memory; -- Error probability -- Ungerboeck codes; -- Computation of power spectra; -- Optimum receiver filter.



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1.0 INTRODUCTION

In the past few years, considerable attention has been devoted to evaluating and optimizing the performance of commercial digital communication systems. Several studies have been devoted to the performance improvement that can be achieved through a more sophisticated design, and the mathematical level of these studies is growing with the complexity of these systems. Thus, the evaluation of their performance becomes an increasingly more demanding task.

The aim of this report is to introduce some new techniques for the analysis and design of digital communication systems. The emphasis here is placed on the computational algorithms that avoid resorting to simulation techniques, which might be either prohibitively expensive, or not completely satisfactory in terms of accuracy.

These digital communication systems that can be studied using the techniques described in the following are: high-rate transmission systems with limited bandwidth; satellite and microwave radio-relay links with nonlinearities; multi-path and fading

transmission systems. The common feature of all these systems, as we shall see in more detail in Section 3, is that they can often be modeled as follows. At the receiver front-end of the system there is a desired signal component disturbed by an additive noise term. Under the assumption that the noiseless portion of the channel has a finite memory (i.e., the received signal at any given time instant does not depend on the infinite past of the transmitted signal) the number of waveforms that can be observed, before the addition of noise, in any time interval of finite duration is finite.

Appendix A includes an overview of some of the techniques for performance evaluation of communication systems. The remainder of this report is devoted to a detailed analysis of some of the techniques that have been studied under this Contract.

2.6 EVALUATION OF ERROR PROBABILITIES

In recent years, there has been an increasing demand for the analysis of digital communication systems, partly due to the availability of large numbers of novel digital signal processing devices and partly due to certain attractive features of digital systems, as for instance the fact that digital signals are more amenable to enciphering and deciphering than analog signals for secure communication purposes. Performance evaluation of these systems is generally based on the average error probability criterion. However, as digital communication systems become more sophisticated, it turns out that many of the known analytical and simulation techniques available for the computation of error probability are either not applicable or prohibitively expensive. This is for example the case of digital communication systems such as: systems operating on a nonlinear bandlimited channel; satellite systems operating in a channelized environment and a nonlinear transponder; spread-spectrum multiple access systems; microwave systems with cochannel interferences. Despite the seemingly diverse nature of these systems, a unifying feature behind all of them is that at their receiver

output there is a desired signal component disturbed by an interference term plus noise. Let the interference term be denoted by X . Then, the error probability for all these systems can be expressed as an expectation of a suitable known function with respect to the random variable X . Usually, this expectation cannot be evaluated directly, either because the statistics of X are not known or because its computation would take too long. Thus, various bounding and approximation techniques have been considered.[1-19]

Basically, these techniques use a set of moments of the random variable X . Ref. [33] includes which is probably the most comprehensive approach to this problem. It describes a class of upper and lower error bounds that can be evaluated with modest computational effort, and can be very tight, particularly when a large number of moments can be evaluated with sufficient accuracy. These bounds can use an arbitrary number of moments of the random interference, and are based on Krein-Nudel'man theory [20-21]. They provide the tightest possible bounds based on a given moment information, and generalize several previously derived theories.

In many situations, however, use of moment bounding techniques cannot be directly applied, either because even the moments cannot be evaluated exactly or because the additive noise does not have a known probability density function. We consider separately these two situations.

2.1 Probability Of Error For Digital Systems With Inaccurately Known Interference

In this section we extend the moment bound theory to the computation of error probabilities to a situation in which the moments of the interference are known only within a given accuracy.

The model assumed for the analysis is the following: consider a digital communication system where the output decision random variable at each sampling instant is given by

$$(2.1) \quad R = ah + X + \nu,$$

where a is the input information symbol taking on values in a finite set, h is a constant, ν is an arbitrary random noise with known cumulative distribution function, and X is the random

interference, which is only known through a set of intervals in which its moments lie.

A situation like this can occur in several instances. For example, the exact statistics of the interference may not be known, and only estimates of the first moments may be available. Another example arises when X represents the intersymbol interference generated by a channel impulse response whose samples are known only in a certain interval. This may occur because those samples have been measured with finite accuracy, or because we want to estimate the error probability for a class of channel impulse responses.

The solution to this problem can be obtained using a theory developed by Krein and Nudel'man in conjunction with the so-called "Chebyshev-Markov problem with moments in a parallelepiped"[21]. This theory deals with the evaluation of upper and lower bounds of the integral

$$(2.2) \quad I = \int_a^b G(x) dP(x)$$

where $G(\cdot)$ is a known function, and $P(\cdot)$ is the cumulative distribution function of a random variable whose first n moments are approximately known.

An outline of the theory of moment bounds with moments in a parallelepiped, as well as the derivation of computational techniques for its solution, are included in Appendix B.

2.2 Probability of Error For Disturbances Known Only Through Their Moments

Another case of interest arises when a signal of known statistics is perturbed by interference and noise, both of whom are known only through their moments. In this case the situation is the following: the observed signal is

$$(2.3) \quad R = ah + X$$

where a is a random variable representing the useful signal and whose probability density function is known; X is a random variable representing the disturbance, and whose first n moments are known. We want to evaluate the probability that R crosses a given threshold t , i.e., the quantity

$$(2.4) \quad G_R(t) = \text{Prob}[R > t]$$

are taken in each waveforms, 2^{18} complex values must be stored, and 1024 256-point Fourier transforms must be evaluated.

For this reason, a particular attention has been devoted to the computational shortcuts that can be devised to reduce the computational complexity and/or the storage requirement of the algorithm. Symmetries arising in the signal set have been taken into account, thus reducing the dimensionality of the signal set by a factor of M in most cases of practical importance. Moreover, the particular structure of the matrix P has been exploited, resulting in an iterative algorithm for the computation of the matrix $\underline{\Lambda}(f)$ in equation (3.17).

The resulting computer algorithm can compute the power spectrum of a digital signal with M up to 4, and a memory L up to 5, in a reasonable time. An example of the results that can be obtained by using this program is reported in Fig.2.

In conclusion, the computation of the power spectrum can be performed through the following steps:

1. Determine the waveforms that are available at the channel output, and their Fourier transforms (use FFT, typically). Arrange them as the components of the vectors $\underline{Q}_k(f)$.
2. Compute the quantities $\underline{c}_2(f)$, $\underline{c}_1(f)$, $c_0(f)$, and $\mu(f)$ from eqs.(3.11)-(3.15)
3. Compute the power density spectrum using (3.16)-(3.18).

Although all these steps are computationally straightforward, Step 1 may put a considerable burden in terms of computer time and storage. In fact, if L denotes the channel memory, and M the source alphabet size, i.e., the number of different source symbols, the number of waveforms to be stored is M^{L+1} . Moreover, each waveform must be represented with an adequate number of samples, and its Fourier transform must be taken. Thus, for example, if $L=4$, $M=4$, and 256 samples

The third quantity is the average amplitude spectrum of the waveforms available from the channel:

$$(3.14) \quad \mu(f) = \underline{w} \underline{c}_2'(f).$$

Finally, the fourth quantity of interest is the average energy spectrum of the waveforms available at the channel output:

$$(3.15) \quad c_0(f) = \sum_{k=1}^M p_k \underline{Q}_k^*(f) \underline{D} \underline{Q}_k'(f)$$

The continuous part and the discrete part of the power density spectrum of the signal $y(t)$ are then given by:

$$(3.16) \quad \underline{Q}_c(f) = \frac{1}{T} [c_0(f) - |\mu(f)|^2] + \frac{2}{T} \Re [\underline{c}_1^*(f) \underline{\Lambda}(f) \underline{c}_2'(f)]$$

and

$$(3.17) \quad \underline{Q}_d(f) = \frac{1}{T^2} |\mu(f)|^2 \sum_{\ell=-\infty}^{\infty} \delta(f - \ell/T)$$

where

$$(3.18) \quad \underline{\Lambda}(f) = \sum_{\ell=1}^L [\underline{p}^{\ell-1} - \underline{p}^{\infty}] e^{-j2\pi f \ell T}.$$

we evaluate four quantities that play the fundamental role in the expression of the power spectral density we are looking for. The first is the average value, taken over the source symbols, of the vectors $\underline{Q}_k(f)$:

$$(3.11) \quad \underline{c}_2(f) = \sum_{k=1}^M p_k \underline{Q}_k(f)$$

The i -th component of $\underline{c}_2(f)$ is then the average Fourier transform of the waveforms available to the channel output when the channel is in state i .

The second quantity is the vector $\underline{c}_1(f)$ whose j -th component is the average Fourier transform of the waveforms that, when output by the modulator, force it to the state j . We have

$$(3.12) \quad \underline{c}_1(f) = \sum_{k=1}^M p_k \underline{Q}_k(f) \underline{D} \underline{E}_k,$$

where

$$(3.13) \quad \underline{D} = \text{diag} [w_1, w_2, \dots]$$

and $[\underline{E}_k]_{ij} = 1$ if the source symbol A_k takes the channel from state i to state j , and zero elsewhere.

We are now in a position to derive the power density spectrum of the digital signal (3.1). Only the final results will be given, as the details can be found in the book [25].

Let us denote by

$$(3.8) \quad w_i = \text{Prob}\{\sigma_n = i\}$$

the stationary state probabilities of the Markov chain, i.e., the components of the unique probability vector w such that

$$(3.9) \quad \underline{w} \underline{P} = \underline{w}.$$

Denote then by $\underline{Q}_k(f)$ the row vector whose entries are the Fourier transforms of the waveforms $q(t; \dots)$, according to the following rule: the i -th component of the vector $\underline{Q}_k(f)$ is the transform of $q(t; A_k, i)$. That is, $\underline{Q}_k(f)$ includes the Fourier transforms of the waveforms corresponding to the source symbol A_k , for the different channel states. Letting also

$$(3.10) \quad p_k = \text{Prob}\{a_n = A_k\}, \quad k=1, \dots, M,$$

As one can see, from state (xyz) the shift register can move only to state (wxy) , with probability p_0 if $w=0$, and p_1 if $w=2$.

Consider then the m -step transition probabilities. These are the elements of the matrix \underline{P}^m . As the channel state depends on L symbols, and these are assumed to be statistically independent, the state σ_{n+m} , $m \geq L$, is independent of the state σ_n , so that we have

$$\begin{aligned} & P\{\sigma_{n+m} = (A_{j_1}, \dots, A_{j_L}) \mid \sigma_n = (A_{i_1}, \dots, A_{i_L})\} \\ (3.6) \quad & = \prod_{l=1}^L P\{a_n = A_{j_l}\}, \quad m \geq L \end{aligned}$$

Thus, $\underline{P}^L = \underline{P}^{L+1} = \dots$, and \underline{P}^L has identical rows. We can write

$$(3.7) \quad \underline{P}^L = \underline{P}^\infty,$$

which shows, in particular, that the channel state sequence (σ_n) is a fully regular Markov chain, i.e., all eigenvalues of the transition probability matrix P having unit magnitude are identically 1, and 1 is a simple eigenvalue.

p_{ij} is given by

$$\begin{aligned}
 p_{ij} &= P\{\sigma_n = (A_{j_1}, \dots, A_{j_L}) | \sigma_{n-1} = (A_{i_1}, \dots, A_{i_L})\} \\
 (3.4) \quad &= P\{a_{n-1} = A_{j_1}, \dots, a_{n-L} = A_{j_L} | a_{n-2} = A_{i_1}, \dots, a_{n-L-1} = A_{i_L}\} \\
 &= P\{a_{n-1} = A_{j_1}\} \delta_{i_1 j_2} \delta_{i_2 j_3} \dots \delta_{i_{L-1} j_L},
 \end{aligned}$$

where δ_{ij} denotes the Kronecker symbol ($\delta_{ii}=1$, and $\delta_{ij}=0$ for $i \neq j$).

As an example, assume $M=2$, $A_1=0$, $A_2=1$, and $L=3$. The noiseless channel has eight states, whose lexicographically ordered set is $\{(000), (001), (010), (111), (100), (101), (110), (111)\}$. The transition probability matrix of the corresponding Markov chain is

$$(3.5) \quad P = \begin{matrix} & \begin{matrix} (000) & (001) & (010) & (011) & (100) & (101) & (110) & (111) \end{matrix} \\ \begin{bmatrix} p_0 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 \\ p_0 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 \\ 0 & p_0 & 0 & 0 & 0 & p_1 & 0 & 0 \\ 0 & p_0 & 0 & 0 & 0 & p_1 & 0 & 0 \\ 0 & 0 & p_0 & 0 & 0 & 0 & p_1 & 0 \\ 0 & 0 & p_0 & 0 & 0 & 0 & p_1 & 0 \\ 0 & 0 & 0 & p_0 & 0 & 0 & 0 & p_1 \\ 0 & 0 & 0 & p_0 & 0 & 0 & 0 & p_1 \end{bmatrix} & \begin{matrix} (000) \\ (001) \\ (010) \\ (011) \\ (100) \\ (101) \\ (110) \\ (111) \end{matrix} \end{matrix}$$

channel states, which forms a Markov chain. If each of the symbols a_n can take on M values, the states can take on M^L values.

To derive the transition matrix of the Markov chain of the channel states, we shall first introduce a suitable ordering for the values of σ_n . This can be done in a rather natural way by firstly ordering the elements of the set $\{A_1, A_2, \dots, A_M\}$ of the M values taken on by the a_n (a simple way of doing that is to stipulate that A_i precedes A_j if and only if $i < j$), and then inducing the following "lexicographical" ordering among the L -tuples $(A_{j_1}, A_{j_2}, \dots, A_{j_L})$:

$$(A_{j_1}, \dots, A_{j_L}) \text{ precedes } (A_{i_1}, \dots, A_{i_L})$$

$$(3.3) \quad \text{if and only if } \begin{cases} j_1 < i_1, \text{ or} \\ j_1 = i_1 \text{ and } j_2 < i_2, \text{ or} \\ j_1 = i_1, j_2 = i_2, \text{ and } j_3 < i_3, \text{ etc.} \end{cases}$$

Once the state set has been ordered according to the rule (3.3), each state can be represented by an integer number expressing its position in the ordered set. Thus, if i represents the state $(A_{i_1}, A_{i_2}, \dots, A_{i_L})$, the one-step transition probability

spectrum. It includes modulator, filters, etc., and transforms the discrete-time input sequence into a continuous waveform $y(t)$. We assume that the channel has a finite memory, i.e., that at time t the channel output $y(t)$ depends only on a finite number, say L , of past source symbols besides the one emitted at time t . We can write

$$(3.1) \quad y(t) = \sum_{n=-\infty}^{\infty} q(t-nT; a_n, \sigma_n)$$

where

$$(3.2) \quad \sigma_n = (a_{n-1}, a_{n-2}, \dots, a_{n-L})$$

is referred to as to the state of the noiseless channel during the n -th time interval $((n-1)T, nT)$. The waveforms $q(t; a_n, \sigma_n)$ take values in a set of deterministic, finite energy signals. The hypothesis that the channel memory is finite justifies the assumption that this set includes a finite number of different waveforms.

The emission of a_n from the source forces a transition of the channel state from σ_n to σ_{n+1} . A sequence of source symbol thus generates a sequence of

An efficient computational technique to evaluate the bandwidth occupancy, and more generally the power spectral density of a digital signal, was developed in [22-24]. This technique is based on a Markov chain model of the digital signal, and is general enough to handle a large variety of situations: for example, it can include the effects of convolutional coding, linear filtering and nonlinear processing of the digital signal.

The communication system to which our theory applies is shown in Fig.1. The source emits a stationary sequence (a_n) of discrete independent

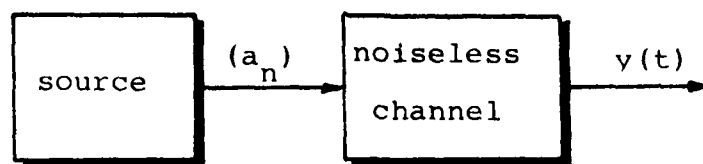


Fig.1 - Noiseless part of a digital communication system.

random variables whose statistics are known. Each a_n is emitted every T seconds. The noiseless channel is assumed to include every device between the source and the point at which we want to evaluate the power

The need for modulation schemes that employ efficiently bandwidth and power in digital radio communication systems has led to the extensive use of phase-shift keying (PSK). As power spectra of PSK signals exhibit sidelobes which may interfere with neighboring channels, a certain amount of filtering is necessary at the transmitter, to provide sidelobes removal. The nonlinearity on the transponder will however affect the power spectrum shape so as to restore the spectral sidelobes, and this effect must be accurately controlled in order to avoid unwanted interference from and to neighboring channels.

For these reasons, it is useful to have a technique to compute the power spectrum spread produced by nonlinearities operating on digital signals. These computations are usually performed by simulation, i.e., by applying a pseudorandom sequence to a model of the system, and taking the Fourier transform of the output signal. Such a procedure, besides being rather time-consuming, usually leads to results showing the power spectrum in a crudely approximated form.

3.0 POWER SPECTRAL DENSITY OF DIGITAL SIGNALS TRANSMITTED OVER NONLINEAR CHANNELS

Efficiency of use of the radio spectrum for both terrestrial and satellite digital systems is a subject of increasing relevance in communications, and spectrum occupancy is a significant parameter in performance evaluation. In fact, in many situations data streams from users are assigned adjacent frequency bands that interfere with each other in a larger or lesser extent depending on the bandwidth occupancy of the modulated signal. Thus, spectrum occupancy is a rough measure of adjacent channel interference.

Moreover, for spectrum conservation the bandwidth occupancy of modulated signals must be kept to a minimum, without impairing the system performance. This is particularly relevant in the presence of channel nonlinearities, which may restore the signal spectrum sidelobes that were previously removed by linear filtering. An important example where this situation occurs is provided by satellite transponders in which a nonlinear device is present -- e.g., a traveling-wave tube amplifier, or a hard limiter.

moments.

The solution of this problem can again be found from Krein-Nudel'man theory of moment bounds [21]. If n moments of the random variable R are known, it is possible to evaluate upper and lower bounds to (2.4) that are sharp, i.e., no other bound based on moments can be tighter. Although the theory is rather involved, the algorithm which provides the result is relatively easy to use. The algorithm itself is based on the search of the roots of a polynomial which provide the points of increase of the distribution functions giving the upper and lower bounds sought. Another algorithm, which appears to be computationally more stable, has been obtained and is presently being developed, and we hope to report on it soon.

More generally, we can assume that we are dealing with a random variable R whose first n moments are known, and we want to find upper and lower bounds to the quantity (2.4) for any given t .

This problem arises in several instances. For example, we may want to determine the cumulative distribution function of a random variable made up as the sum of a number of independent random variables. The exact distribution function, apart from a few special cases, is difficult to determine, but as the moments of a sum of random variables are relatively easy to compute, we can use this theory to find upper and lower bounds to the distribution function. This works better than the assumption, which is often made in this situation, that the random variable we are dealing with are Gaussian -- this may make sense when the number of random variables involved is very large, but has still to be verified. Yet another application arises when a digital communication system is perturbed by a noise whose exact statistics are not known. Sometimes the assumption of a given distribution whose parameters fit those measured will work, but it may be preferable to use the theory presented here, which requires no hypotheses to be added to the knowledge of

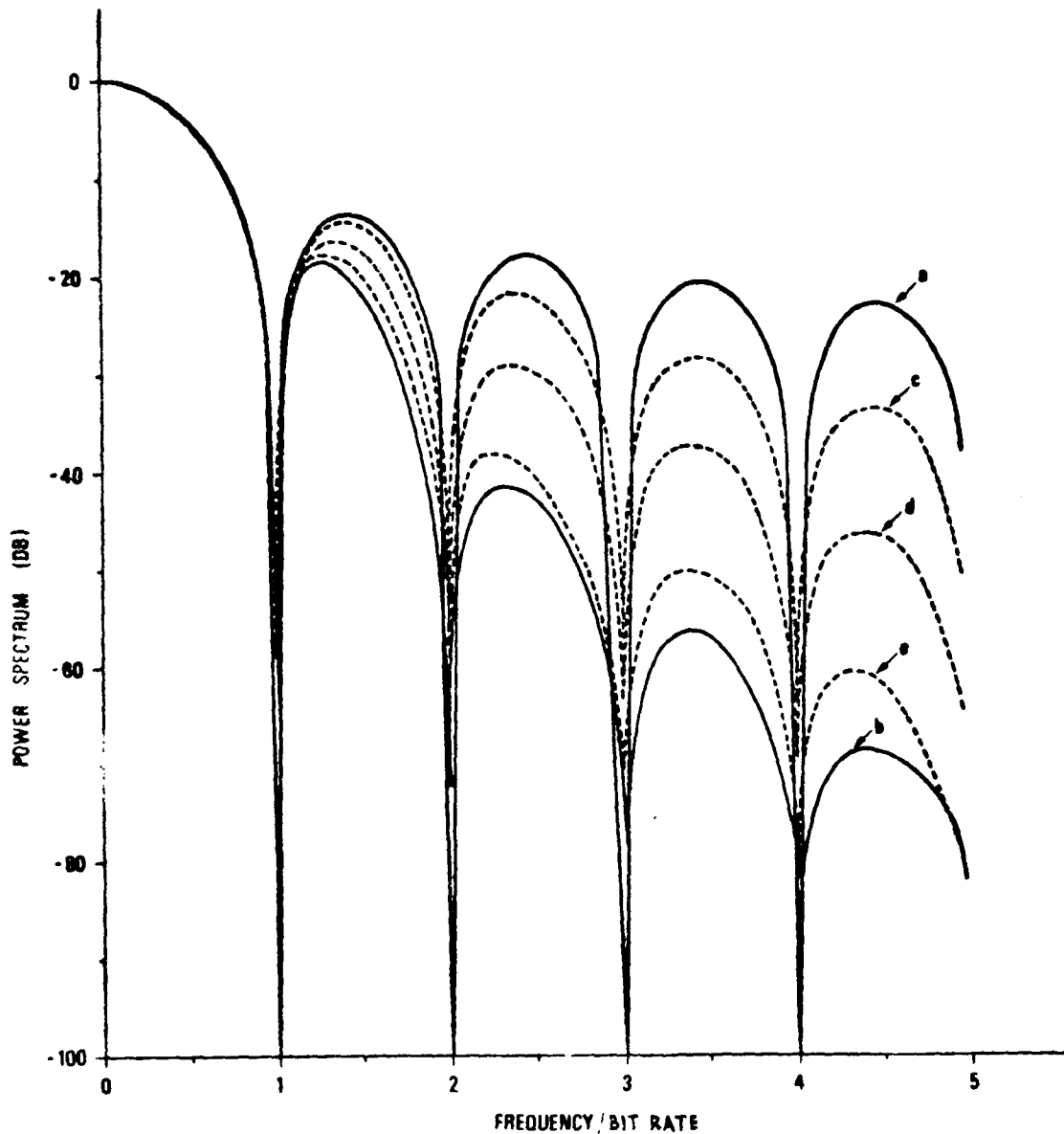


Fig.2 - Power density spectrum of a binary PSK signal.
 a- Before filtering
 b- After filtering (filter is a 4-pole Butterworth with a 3-dB bandwidth $2.4 R$, R the bit rate)
 c- After a TWT driven at saturation
 d- After the same TWT, with a 6-dB input backoff
 e- After the same TWT, with a 12-dB input backoff.

This figure shows the power density spectrum of a binary PSK signal first filtered through a fourth-order Butterworth filter , and then passed through a nonlinear amplifier driven at or near saturation. It is seen that, although filtering removes the sidelobes, they are restored by the nonlinearity, to an amount that can be evaluated with a good accuracy.

Other avenues of attack have been explored as well. One which at first seemed particularly promising is the use of Volterra series to model the nonlinear channel. This approach proved to be very fruitful in the context of evaluation of error probabilities, as explored in [26-27]. However, Volterra series do not appear to offer any particular advantage when the nonlinear channel has to be modeled in order to derive the power density spectrum. Actually, our practice has shown that when the channel is made up by cascading a number of blocks (typically, filters and nonlinear memoryless devices) the best approach to derive the waveforms at its output is simulation. After this preliminary simulation has been performed, the analytical tools previously described can be used to compute the power spectrum.

4.6 OPTIMAL RECEIVING FILTER FOR DIGITAL TRANSMISSION OVER NONLINEAR CHANNELS

A problem arising in digital transmission over nonlinear channels is the design of modem filters. Filters incorporated in present-day modems are not necessarily optimum when the channel is nonlinear, as they are usually designed with the Nyquist theory in mind. The early INTELSAT 4-PSK TDMA modems specified Nyquist filtering, with a 30-percent rolloff Nyquist transmitting filter combined with an $f/\sin f$ aperture equalizer. The receiving filter was a high-order elliptic filter, whose bandwidth was selected so as to maximize the noise rejection while not degrading the Nyquist response of the transmitting filter [28]. Recent research is aimed at selecting optimum filters for a nonlinear channel model, either by choosing the best values of the parameters within a given filter family [29] or by designing the frequency characteristics of a filter which is optimum under a specified criterion [30].

By using the channel model described in the previous Section of this Report, and which is based on a set of output waveforms connected to form a Markov chain, we developed a theory to design an optimum

receiving filter for digital transmission over a nonlinear channel. The filter is chosen so as to minimize the mean-square error between the transmitted symbols and the samples of the demodulated waveform. Besides providing closed-form expressions for such optimal filters, it was shown that the structure of the filter corresponds to a bank of matched filters, each followed by a linear transversal filter. This result is a generalization of a well-known property of optimum receiving filters for linear channels. The performance of the optimum receivers can be computed, and its performance compared against that obtained through conventional designs.

Some results obtained are reported in Appendix C. It must be noticed that the full generality allowed by the theory developed there has not been exploited in full. We hope to report soon on this topic, by preparing a comprehensive paper covering also the connection to previously known theories, and showing the applicability of our theory to a situation in which the channel disturbances include also interference from adjacent channels.

5.6 MULTIDIMENSIONAL MODULATION AND CODING FOR BANDLIMITED CHANNELS

In digital communication over radio channels, both available spectrum and transmitter power are generally limited. Thus, to cope with the ever-increasing demand for digital communication services, more efficient transmission techniques are called for, and the search for bandwidth- and power-efficient modulation systems has become a very active research area. In fact, it has been recognized that a third relevant factor enters in the tradeoff between bandwidth and power, that is, the complexity of the communication system. In other words, if a certain amount of signal processing is allowed at the transmitting and receiving ends of the system, its performance can be improved without increasing neither its bandwidth nor its power.

For example, as higher-dimensional signal sets are known to afford possible performance improvements, four-dimensional modulation can be used in the signal space constituted by two orthogonally polarized electromagnetic waves. Another technique, which was recently proposed by Ungerboeck [31] and which has been receiving a wide attention (see also [32]), is based on a combined modulation/coding approach. In standard

applications of error-correcting codes, extra bits must be added to the transmitted symbol sequence, with the modulator operating at a higher rate, and hence requiring a larger bandwidth. On the other hand, use of coding can decrease the power requirement necessary to achieve a given performance, so that this gain in performance (usually referred to as the "coding gain") can compensate for the sacrificed bandwidth. In other words, the tradeoff is increased complexity and decreased bandwidth efficiency for increased power efficiency. With Ungerboeck codes, one can avoid the loss in bandwidth efficiency by using a technique characterized by the fact that the redundancy required by the coding process is provided by increasing the number of coded symbols instead of the bandwidth. With these codes, the tradeoff is increased complexity for increased power efficiency.

Some of the results we obtained in the study of this class of codes are included in Appendix D and Appendix E. In particular, we considered a combination of multidimensional signal sets and Ungerboeck codes, an idea which seems to be very promising in terms of applications. We introduced a new class of signal sets for modulation, that we call "generalized group codes"

and which are based on a peak-energy constraint. These alphabets have a good deal of symmetry, a feature which is apparently necessary to design good Ungerboeck codes. Moreover, design techniques are derived, based on a partition of the signal set stemming from the partition of a group into cosets of a convenient subgroup (Appendix D).

We have also investigated the power spectral density properties of the signals obtained from an Ungerboeck code. In particular, we have shown rigorously that, under certain mild symmetry constraints, Ungerboeck codes do not alter the power spectrum of the modulated signals (Appendix E).

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APPENDIX

In this Appendix we compute the probabilities $P\{\xi_{k+1}=j|\xi_k=i\}$ for the situation described in Section 1. We assume that the random variables in the sequence (a_n) of data symbols are independent, and take on values in the set $A=\{\alpha_0, \alpha_1, \dots, \alpha_{N-1}\}$ with probabilities $p_k = P\{a_n = \alpha_k\}$. It can be seen that the channel states form a homogeneous Markov chain. The one-to-one correspondence between channel states and the integers $1, \dots, M$ can be set by ordering the elements of A (for example, $\alpha_0 < \alpha_1 < \dots < \alpha_{N-1}$) and introducing among the $(L+1)$ -tuples $(a_n, a_{n-1}, \dots, a_{n-L})$ the lexicographical order induced by the order of A . The states can then be represented by integer numbers from 1 to M by expressing their position in the ordered set. Thus, if i represents the state $(\alpha_{i_0}, \alpha_{i_1}, \dots, \alpha_{i_L})$ and j the state $(\alpha_{j_0}, \alpha_{j_1}, \dots, \alpha_{j_L})$, we have

$$\begin{aligned} P\{\xi_{k+1}=j|\xi_k=i\} &= P\{a_{k+1}=\alpha_{j_0}, a_k=\alpha_{j_1}, \dots, a_{k-L+1}=\alpha_{j_{L-1}} | a_k=\alpha_{i_0}, \dots, a_{k-L+1}=\alpha_{i_{L-1}}\} \\ &= P\{a_{k+1}=\alpha_{j_0}, a_k=\alpha_{j_1}, \dots, a_{k-L+1}=\alpha_{j_{L-1}} | a_k=\alpha_{i_0}, \dots, a_{k-L+1}=\alpha_{i_{L-1}}\} \\ &= p_{j_0} p_{i_0} p_{j_1} p_{i_1} p_{j_2} p_{i_2} \dots p_{j_{L-1}} p_{i_{L-1}} \end{aligned}$$

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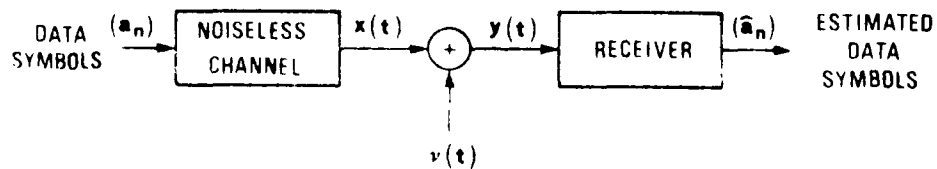


Fig. 1 - Model of a digital communication system.

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Assume that we want to evaluate the power spectrum of the signal $x(t)$ at the output of the channel of Fig.1. If $\underline{Q}(f)$ denotes the vector of Fourier transforms of the waveforms $\{q(t;i)\}_{i=1}^M$, and the symbols a_n are independent and identically distributed, with the simplifying assumption

$$\sum_{i=1}^M q(t;i) = 0$$

we have the following result for the power spectrum of $x(t)$:

$$G_x(f) = \frac{1}{MT} \operatorname{Re} \{ \underline{Q}^{\dagger}(f) \underline{A}(f) \underline{Q}(f) \}$$

where $\operatorname{Re}(\cdot)$ denotes the real part, the dagger denotes conjugate transpose,

$$\underline{A}(f) = \underline{I} + p \underline{P} e^{-j2\pi fT} + p^2 \underline{P}^2 e^{-j4\pi fT} + \dots + p^L \underline{P}^L e^{-j2L\pi fT}$$

and \underline{P} is the $M \times M$ matrix whose elements are the probabilities

$$P_{ij} \triangleq P(\xi_{i+1} = j | \xi_i = i)$$

(see the Appendix).

6.0 CONCLUSIONS

We have described a number of parameters (error probability, minimum distance, cutoff rate, spectrum occupancy) that are useful to evaluate the performance of a digital communication system. A rather general system model has been assumed, which is easily amenable to analysis and is based on the hypothesis of a finite memory for the transmission channel. Based on this model, we have described the computational techniques that can be applied for evaluating those parameters. [14]

$$a_{(i,j),(k,m)} \triangleq P\{\xi_{\ell+1} = k | \xi_{\ell} = i\} P\{\xi'_{\ell+1} = m | \xi'_{\ell} = j\} \cdot \exp\left(-\frac{1}{4N_0} d^2(k,m)\right) \quad (4.5)$$

where

$$d^2(i,j) \triangleq \int_0^T |q(t;i) - q(t;j)|^2 dt$$

are the Euclidean distances between the waveforms of the set $\{q(t;i)\}_{i=1}^M$. The probabilities appearing in (4.3) can be computed as shown in the Appendix.

5.0 COMPUTATION OF THE POWER SPECTRUM

Efficient use of the radio spectrum is a subject of increasing relevance in communication. For spectrum conservation the bandwidth occupancy of any communication system must be kept to a minimum without impairing its performance. As many channels as possible should be accommodated in a given portion of the frequency spectrum. As a consequence, the spectrum occupancy turns out to be a significant parameter in the design of a transmission system in which several users share a given frequency band. Whatever the criterion to evaluate the spectrum occupancy (for example, the bandwidth in which 99.99% of the signal power is contained) its computation must be based on the power spectrum of a digital signal. The computation of power spectrum is a relatively simple task when performed by simulation, which is done by applying a pseudorandom data sequence at the system input and computing the Fourier transform of the output signal. However, this procedure, besides being rather time-consuming, leads usually to the power spectrum in a crudely approximated form. A computationally efficient technique, which combines simulation and mathematical analysis, is described in the sequel.

is described in [10].

4.0 COMPUTATION OF THE CUTOFF RATE

Consider now a digital communication system in which a code has to be used. To evaluate the coding capabilities of the channel, a reasonable criterion would be to choose the best code for that channel, and the corresponding error probability. But this cannot be done in general, because the selection of the best code is hardly feasible in the practice. Now, information theory provides us with a result concerning the error probability that can be achieved over a given channel as a function of the transmission rate. The result is the following : there exists a block code of rate R and length n such that the error probability is upper bounded by

$$p(e) \leq c e^{-n(R - R_0)} \quad R < R_0 \quad (4.1)$$

where c is a constant, and R_0 is a parameter depending on the channel, and called its cutoff rate. This relation places in evidence the way at which the attainable error probability decays as a function of the block length of the code. The larger R_0 , the better the channel. Hence, R_0 provides a useful one-parameter characterization of the quality of the channel in terms of its coding capabilities [11].

An expression for the cutoff rate of the channel of Fig.1 has been derived in [12] (see also [13] for similar computations when a convolutional code is used). The following result holds:

$$R_0 = -\log_2 \lambda \quad (4.2)$$

where λ is the unique real and positive eigenvalue of the $M^2 \times M^2$ matrix whose elements are

(3.1) it is seen that this is the relevant parameter for assessing the performance of a digital system equipped with a maximum-likelihood sequence detector.

Unfortunately, the minimum distance does not seem to be expressible in a closed form. Moreover, its direct computation based on (3.2) is impractical in most cases of interest. Thus, for the evaluation of d_{\min} one must resort to algorithms suitably designed. One such algorithm has been applied in [8] to the transmission over linear channels. It is based on a computer-search approach, and results into a small "sufficient" set of sequence pairs (ξ_n, ξ'_n) differing in few positions. The minimum of $\|\sum_n [q(t-nT; \xi_n) - q(t-nT; \xi'_n)]\|$ taken in this set provides the minimum distance.

The computation of d_{\min} for a nonlinear channel has been studied in [9], under the simplifying hypothesis that only symbol sequences differing in just one symbol contribute to the value of the minimum distance. Consider for simplicity binary symbols, and $L+1$ consecutive channel states $\Xi \triangleq (\xi_n, \xi_{n+1}, \dots, \xi_{n+L})$. Take another $L+1$ -tuple $\Xi' \triangleq (\xi'_n, \xi'_{n+1}, \dots, \xi'_{n+L})$, with ξ'_n differing from ξ_n only in the first position, ξ'_{n+1} differing from ξ_{n+1} only in the second position, ξ'_{n+2} from ξ_{n+2} in the third, etc. Compute then

$$d'(\Xi, \Xi') = \sum_{i=0}^L \|q(t; \xi'_{n+i}) - q(t; \xi_{n+i})\|^{1/2}$$

for all possible pairs Ξ, Ξ' (for uncoded binary symbols, there are $2^{2(L+1)}$ such pairs) and take the minimum value found. This is the minimum distance. Although this method needs the manipulation of a large number of sequences, the numerical operations required are quite simple.

Yet another technique for the computation of the minimum distance

same set of moments can be tighter.

Moment bound techniques are described in [3] for linear channels, and in [1],[2],[4] for nonlinear channels. In [5] the case of correlated random variables a_n is considered.

3.0 COMPUTATION OF THE MINIMUM DISTANCE

In the previous section we have considered the error probability of a simple receiver whose decisions are taken symbol-by-symbol. In certain systems, however, it may prove highly beneficial to include a more sophisticated receiver to achieve nearly ideal performance. This can be done by using a maximum likelihood receiver, in which the decisions on a data sequence are taken by considering all the possible transmitted sequences and choosing the one most likely on the basis of the observed signal $y(t)$. The operation of such a receiver is analyzed in [6] for linear channels, and in [7] for a nonlinear satellite channel.

When this receiver is used, the symbol error probability is closely approximated, for a small noise power, by

$$P(e) \approx K \operatorname{erfc} \left(\frac{d_{\min}}{2\sqrt{N_0}} \right) \quad (3.1)$$

where K is a constant, $N_0/2$ is the two-sided power spectral density of the noise $v(t)$, and d_{\min} is the minimum distance, defined as

$$d_{\min}^2 \triangleq \min_{(\xi_n') \neq (\xi_n)} \left\| \sum_n \{q(t-nT; \xi_n') - q(t-nT; \xi_n)\} \right\|^2 \quad (3.2)$$

with $\|z(t)\|^2$ denoting the energy of the signal $z(t)$. In words, d_{\min} is the smallest possible Euclidean distance attainable between received signals stemming from different transmitted symbol sequences. From

$$P(\cdot) = \frac{1}{2} E \left\{ \operatorname{erfc} \left[\frac{h_0 + \cdot}{\sqrt{\sigma_{0_v}}} \right] \right\} \quad (2.1)$$

If the channel is not linear, $q(t_0; \xi_n)$ depends on $a_n, a_{n-1}, \dots, a_{n-L}$ in a nonlinear way. A useful representation of this dependence is provided by expanding the functional relationship between $q(t_0; \xi_n)$ and $a_n, a_{n-1}, \dots, a_{n-L}$ in a Volterra series, which will be truncated to a finite number of terms. This representation is discussed in [1] for baseband channels, and in [2] for passband channels.

Consider now the problem of evaluating the averages appearing in (2.2). In the most general case, we are faced with the computation of an average of the form $E[f(a_{n-1}, a_{n-2}, \dots, a_{n-L})]$, where $f(\cdot)$ is a known function of the L random variables a_{n-1}, \dots, a_{n-L} . In principle, we can compute this average by enumerating all the possible values taken on by $f(\cdot)$, but this technique is computationally impractical whenever the number of these values is very large. As an example, if the random variables a_n are independent and take on N values, there are N^L possible values for $f(\cdot)$. For instance, $N=8$ and $L=10$ would lead to about 10^9 computations of the function $f(\cdot)$.

A method that yields both accurate and computationally tractable results for the computation of the average required is based upon the theory of moment bounds. Basically, it consists in the derivation of upper and lower bounds to $E[f(a_{n-1}, \dots, a_{n-L})]$ based on the exact evaluation of a number of averages like $E[p(a_{n-1}, \dots, a_{n-L})]$, where $p(\cdot)$ is a polynomial function. When the random variables a_n are statistically independent, the bounds can be expressed in terms of the moments

$$\mu_k = E\{a_n^k\} \quad (2.7)$$

These bounds are optimum, in the sense that no other bounds using the

$$\begin{aligned}
P(e) &= P\{\hat{a}_n \neq a_n\} \\
&= \frac{1}{2} P(y(t_n) > 0 | a_n = -1) + \frac{1}{2} P(y(t_n) < 0 | a_n = +1) \\
&= \frac{1}{2} P(q(t_0; \xi_n) + v > 0 | a_n = -1) + \frac{1}{2} P(q(t_0; \xi_n) + v < 0 | a_n = +1)
\end{aligned} \tag{2.1}$$

where $v \triangleq v(t_0 + nT)$ is a Gaussian random variable with zero mean and known variance σ_v^2 . From (2.1) we can write

$$\begin{aligned}
P(e) &= \frac{1}{4} E\left\{\operatorname{erfc}\left(\frac{-q(t_0; \xi_n)}{\sqrt{2} \sigma_v}\right) | a_n = -1\right\} \\
&\quad + \frac{1}{4} E\left\{\operatorname{erfc}\left(\frac{q(t_0; \xi_n)}{\sqrt{2} \sigma_v}\right) | a_n = +1\right\}
\end{aligned} \tag{2.2}$$

where E denotes average taken with respect to the random variable ξ_n .

The following tasks must be accomplished for the computation of $P(e)$. Firstly, we have to write down an explicit expression for $q(t_0; \xi_n)$. Secondly, we must compute the averages in (2.2).

Suppose first that the channel is linear. Then $q(t_0; \xi_n)$ depends linearly on the random variables $a_n, a_{n-1}, \dots, a_{n-L}$, so we can write

$$q(t_0; \xi_n) = \sum_{i=0}^L h_i a_{n-i} \tag{2.3}$$

where h_0, h_1, \dots, h_L are the samples of the impulse response of the channel. Hence

$$q(t_0; \xi_n) = h_0 a_n + Z \tag{2.4}$$

where

$$Z \triangleq \sum_{i=1}^L h_i a_{n-i} \tag{2.5}$$

is called the intersymbol interference term. Observing that Z is a symmetric random variable, i.e., Z and $-Z$ are equally distributed, we have from (2.2):

where $\{q(t;i)\}_{i=1}^M$ is the set of waveforms (of duration T) observable at the output of the noiseless channel in the interval $(0,T)$. The sequence $(\xi_n)_{n=-\infty}^{\infty}$ is made up of random variables, called the channel states, taking on M possible values. With our assumption of a finite memory for the noiseless channel, each ξ_n depends on a finite number, say $L+1$, of input data symbols a_n , so that we can set a one-to-one correspondence among the ξ_n and the $(L+1)$ -tuples $(a_n, a_{n-1}, \dots, a_{n-L})$. This correspondence allows one to determine the statistics of the sequence (ξ_n) , and hence of $x(t)$, when the statistics of the data sequence (a_n) are completely known. We assume here that (a_n) is stationary.

The signal observed at the receiver front-end is, from Fig.1,

$$y(t) = x(t) + v(t) \quad (1.2)$$

where $v(t)$ is assumed to be a white Gaussian noise process.

2.0 COMPUTATION OF ERROR PROBABILITY

The error probability is a basic measure of the performance of digital communication systems. In this section we consider a situation in which the receiver operates by sampling the received signal $y(t)$ every T sec. The decision \hat{a}_n on the n -th transmitted symbol is based only on the value of this sample. For simplicity's sake, we shall confine ourselves to consideration of the case in which the input data symbols take on with equal probabilities the values ± 1 , and modulate linearly a given waveform (this situation corresponds to binary PAM, or PSK). The receiver compares the received sample with a zero threshold, and sets $\hat{a}_n = +1$ if it lies above the threshold, $\hat{a}_n = -1$ otherwise. For a sample taken at time $t_n = t_0 + nT$, the symbol error probability is given by

the evaluation of their performances becomes more demanding. For example, in satellite communications the efficient use of available signal power and bandwidth makes them to operate on a tightly bandlimited nonlinear channel, where the computation of error rates (say) is not analytically tractable.

The aim of this paper is to review some of the techniques that have been recently proposed for evaluating the performance of digital communication systems. The emphasis is placed here on the computational algorithms that allow this evaluation to be performed without resorting to simulation techniques, which might be either prohibitively expensive or not completely satisfactory in terms of accuracy.

We are interested in considering digital communication systems such as: high data-rate transmission systems with limited bandwidth; satellite and microwave radio-relay links with nonlinearities; multi-path and fading transmission systems. The common feature of all these systems is that they can often be modeled as follows. At the receiver front-end of the system there is a desired signal component disturbed by an additive noise term. Under the assumption that the noiseless portion of the channel has a finite memory (i.e., the received signal at any given time instant does not depend on the infinite past of the transmitted signal) the number of waveforms that can be observed, before the addition of noise, in any time interval of finite duration is finite. With reference to Fig.1, if T denotes the inverse of the symbol rate, i.e., the time interval between the emission of two consecutive symbols, the signal $x(t)$ can be represented in the form

$$x(t) = \sum_{n=-\infty}^{\infty} a_n p(t-nT; \xi_n) \quad (1.1)$$

COMPUTATIONAL TECHNIQUES FOR EVALUATION OF COMMUNICATION SYSTEM PERFORMANCE

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SUMMARY

In this paper we review some techniques for evaluating the performance of digital communication systems operating on channels characterized by additive Gaussian noise as well as linear and nonlinear distortions. The parameters considered are the error probability, the minimum distance (useful when a maximum-likelihood sequence receiver is used), the cutoff rate (useful when coding has to be used on the channel), and the spectral occupancy (useful when two or more users share the same frequency band). The emphasis is placed on the computational algorithms that allow these parameters to be evaluated numerically.

1.0 INTRODUCTION AND MOTIVATION OF THE WORK

In recent years, there has been an increasing interest in digital communication systems. This is partially due to the availability of novel digital signal processing devices as well as to certain attractive features of digital systems, as for instance the fact that digital signals are more amenable to enciphering and deciphering than analog signals for secure communication purposes. On the other hand, the ever-increasing demand for digital services has suggested the introduction of more efficient, and hence more sophisticated, communication systems. As the complexity of these systems increases,

Appendix A

12. E.Biglieri, "Analysis of random digital signals", XXIst General Assembly of URSI, Florence (Italy), August-September 1984
13. E.Biglieri and M.Elia, "Multidimensional modulation and coding for bandlimited digital channels", International Conference on Computers, Systems and Signal Processing, Bangalore (India), December 1984

Appendix B

measured with finite accuracy, or because we want to estimate the error probability for a class of channel impulse responses. Several numerical examples are provided which show the range of applicability of this technique.

I. INTRODUCTION

In recent years, several techniques have been proposed to evaluate error probabilities for digital communication systems in the presence of additive noise and random interference. Among these techniques, multidimensional moment bounds [1] appear to be most useful, because they are generally very tight and can be evaluated with modest computational effort. Moreover, no other bounds based only on the moments of the random interference can be tighter.

In this correspondence we extend the moment bound theory to the computation of error probabilities in a situation in which the moments of the interference are known only in certain intervals. The model assumed for the analysis is the following: consider a digital communication system where the output decision random variable at each sampling instant is given by

$$R = \alpha h + Z + \nu, \quad (1.1)$$

where α is the input information symbol taking on values in a finite set, h is the known positive peak overall sampled system response, ν is an arbitrary random noise with known cumulative distribution function $F_\nu(\cdot)$, and Z is the random interference, typically modeled as a sum of independent random variables. We assume that the random variables α , Z , and ν are mutually independent, and that Z has a finite range $[-D, D]$. Evaluation of the error probability of the above system can be based on moment bound theory by first expressing it as the average

$$P_e = E_Y[\Omega(Y)], \quad (1.2)$$

where $Y = f(Z)$ is a random variable with finite range $[a, b]$, $\Omega(\cdot)$, $f(\cdot)$ are known functions, and E_Y denotes average taken with respect to the random variable Y . Then, if the finite sequence of moments $\{c_i\}_{i=1}^n$ is available, where

$$c_i \triangleq E[Y^i], \quad i = 1, \dots, n, \quad (1.3)$$

the moment bound theory, as developed by Krein [2], provides the tightest upper and lower bounds to $E_Y[\Omega(Y)]$ in the form

$$\sum_{i=1}^{N'} w_i' \Omega(y_i') \leq E_Y[\Omega(Y)] \leq \sum_{i=1}^{N''} w_i'' \Omega(y_i''), \quad (1.4)$$

where the abscissas $\{y_i'\}_{i=1}^{N'}$, $\{y_i''\}_{i=1}^{N''}$ and the weights $\{w_i'\}_{i=1}^{N'}$, $\{w_i''\}_{i=1}^{N''}$ can be computed on the basis of the moment set $\{c_i\}_{i=1}^n$. The values of N' and N'' , as well as the rules for computing weights and abscissas in (1.4), vary according to the parity of n and to the sign of the $(n+3)$ rd derivative of the function Ω in the interval $[a, b]$. Details on the actual evaluation of moment bounds can be found in [1], together with a number of generalizations of the model considered here.

As an example, if α takes on values ± 1 with equal probabilities, and ν is a zero-mean Gaussian random variable with variance σ^2 , the error probability in (1.2) takes the form

$$P_e = E_Y \left[Q \left(\frac{h + Z}{\sigma} \right) \right], \quad (1.5)$$

where

$$Q(x) \triangleq (2\pi)^{-1/2} \int_x^\infty \exp(-t^2/2) dt.$$

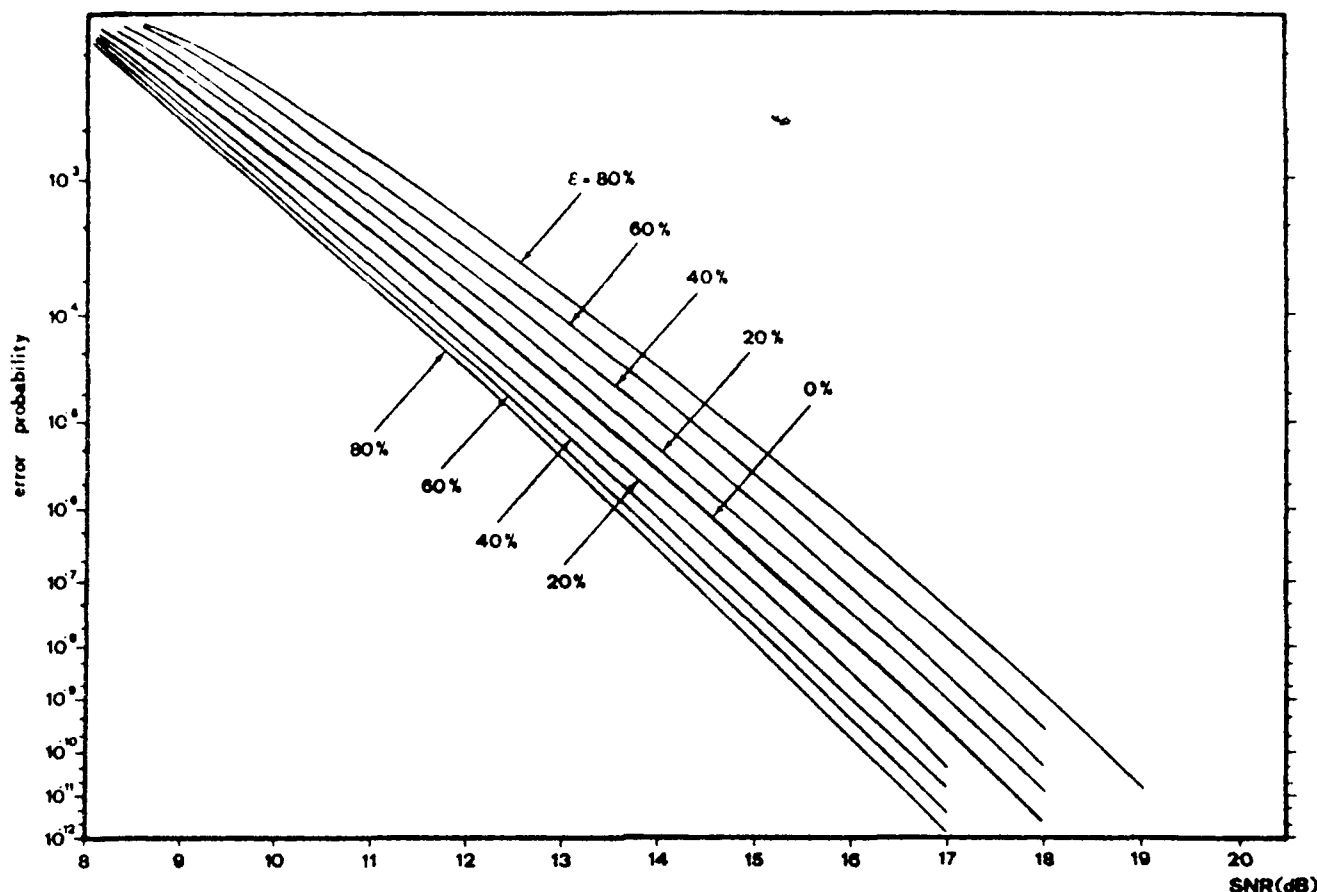
Probability of Error for Digital Systems with Inaccurately Known Interference

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Abstract—The "moment bound theory" is known to provide a useful technique to evaluate error probabilities for digital communication systems in the presence of additive noise and random interference. In this correspondence this theory is extended to the case where the moments of the interference are known only within certain intervals, and upper and lower bounds to the error probabilities are sought. A situation like this can occur in several applications. For example, the exact statistics of the interference may not be known, and only estimates of the first moments may be available. Another example arises when the signal is disturbed by intersymbol interference generated by a channel impulse response whose samples are known only in a certain interval—either because they have been

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Fig. 1. Locus Π of vectors c such that $c' \leq c \leq c''$.

Hence, we can write

$$P_e = E_Y \left[Q \left(\frac{Y}{\sigma} \right) \right]. \quad (1.6)$$

This is equivalent to the choice

$$Y = h + Z \quad (1.7)$$

and $\Omega(x) = Q(x/\sigma)$ in (1.2). However, this is by no means the only available choice. In fact, if we take

$$Y = (h + Z)^2, \quad (1.8)$$

the error probability can be expressed as

$$P_e = E_Y \left[Q \left(\frac{\sqrt{Y}}{\sigma} \right) \right], \quad (1.9)$$

which is valid provided that $h - D \geq 0$ (the "open-eye" assumption). In this case $\Omega(x) = Q(\sqrt{x}/\sigma)$.

In this correspondence we shall consider the situation in which the moments $(c_i)_{i=1}^n$ are not available exactly, but it is known that they lie in finite intervals. In other words, two sets $(\underline{c}_i)_{i=1}^n$ and $(\bar{c}_i)_{i=1}^n$ are known such that

$$\underline{c}_i \leq c_i \leq \bar{c}_i, \quad i = 1, \dots, n, \quad (1.10)$$

and we want to find upper and lower bounds to the system error probabilities.

A situation like this can occur in several applications. For example, the exact statistics of the interference may not be known, and only estimates of the first moments may be available. Another example arises when Z represents the intersymbol interference generated by a channel impulse response whose samples are known only in a certain interval. This may occur because those samples have been measured with finite accuracy, or be-

cause we want to estimate the error probability for a class of channel impulse responses.

The solution to this problem can be obtained by using a theory developed by Krein in conjunction with the so-called "Čebyšev-Markov problem with moments in a parallelepiped" [2]. This theory deals with the evaluation of upper and lower bounds of the integral

$$I = \int_a^b G(x) dP(x),$$

where $G(\cdot)$ is a known function, and $P(\cdot)$ is the cumulative distribution function of a random variable whose first n moments are specified in the form (1.10).

II. MOMENT BOUNDS WITH MOMENTS IN A PARALLELEPIPED

In this section we shall briefly outline the theory of moment bounds with moments in a parallelepiped, and we shall derive a solution to the problem posed in Section I.

Consider first the set of all the probability distribution functions $F_Y(\cdot)$ of the random variable Y whose range is the finite interval $[a, b]$ and whose first n moments are c_1, \dots, c_n . Denoting such set by $V(c)$, where c is the n -vector whose components are c_1, \dots, c_n , the "classical" moment problem can be formulated as the search for the minimum and maximum values of the integral $\int_a^b \Omega(y) dF_Y(y)$ as F_Y ranges over $V(c)$. More specifically, we seek

$$I(c) = \min_{F_Y \in V(c)} \int_a^b \Omega(y) dF_Y(y), \quad (2.1)$$

and

$$J(c) = \max_{F_Y \in V(c)} \int_a^b \Omega(y) dF_Y(y). \quad (2.2)$$

Consider then two n -vectors c' and c'' , with components c'_1, \dots, c'_n and c''_1, \dots, c''_n , respectively. We define a partial ordering in the space \mathbb{R}^n of n -vectors by defining $c' \leq c''$ if $c'_k \leq c''_k$ and

$$(-1)^{n-k} c'_k \leq (-1)^{n-k} c''_k, \quad k = 1, \dots, n. \quad (2.3)$$

The set of vectors c such that $c' \leq c \leq c''$ is then the parallelepiped Π

$$(-1)^{n-k} c'_k \leq (-1)^{n-k} c_k \leq (-1)^{n-k} c''_k, \quad k = 1, \dots, n. \quad (2.4)$$

The points c' and c'' are the endpoints of the "oblique diagonal" of Π (see Fig. 1 for an example). We have the following result.

Theorem 1 [2, p. 208]: If c' and c'' are moment sequences of random variables with finite range $[a, b]$, and $a > 0$, then any c such that $c' \leq c \leq c''$ is a moment sequence.

Let us now suppose that the random variable Y has moment sequence lying in the parallelepiped Π whose endpoints of the oblique diagonal are the n -vectors $(c'_1, c'_2, c'_3, \dots)$ and $(c''_1, c''_2, c''_3, \dots)$. If we denote by $V(\Pi)$ the set of all the probability density functions whose range is $[a, b]$ and whose moment sequence lies in the Π , the moment problem considered in the introduction can be formulated as the search for the minimum and maximum values of the integral $\int_a^b \Omega(y) dF_Y(y)$ as $F_Y(\cdot)$ ranges over $V(\Pi)$. That is

$$I(\Pi) = \min_{F_Y \in V(\Pi)} \int_a^b \Omega(y) dF_Y(y), \quad (2.5)$$

and

$$\bar{I}(\Pi) = \max_{F_Y \in V(\Pi)} \int_a^b \Omega(y) dF_Y(y). \quad (2.6)$$

The following theorem holds.

Theorem 2 [2, p. 220]: If Π is the set of n -vectors c such that $c' \leq c \leq c''$, then

$$I(\Pi) = I(c') \quad (2.7)$$

and

$$\bar{I}(\Pi) = \bar{I}(c'') \quad (2.8)$$

provided that

$$a > 0 \quad (2.9)$$

$$(-1)^{n+1-k} \Omega^{(k)}(a) \geq 0, \quad k = 1, 2, \dots, n, \quad (2.10)$$

$$\Omega^{(n+1)}(t) > 0, \quad \text{for } a \leq t \leq b. \quad (2.11)$$

As $a > 0$ may be arbitrarily small, the rule for determining $I(\Pi)$ and $\bar{I}(\Pi)$ embodied in Theorem 2 can be extended to the case $a = 0$ [2, p. 220].

Before proceeding further, let us comment briefly on the results summarized by the theorem. If the technical conditions (2.9)–(2.11) are satisfied, then $E[\Omega(Y)]$, and hence the error probability, is bounded above and below by two "standard" moment bounds $I(c')$ and $\bar{I}(c'')$. Evaluation of these two bounds can be performed by using the techniques described in [1], where the moment sequences to be used for the calculation are obtained by taking alternately the upper and lower bounds of (1.10). In fact, by comparison of (1.10) and (2.4) it is seen that the vectors c' and c'' to be used in (2.7)–(2.8) are given by

$$c' = (c'_1, c'_2, c'_3, c'_4, \dots, c'_n)$$

$$c'' = (c''_1, c''_2, c''_3, c''_4, \dots, c''_n)$$

for n even, and

$$c' = (c'_1, c'_2, c'_3, c'_4, \dots, c'_n)$$

$$c'' = (c''_1, c''_2, c''_3, c''_4, \dots, c''_n)$$

for n odd.

TABLE I
CONDITIONS FOR THE VALIDITY OF THEOREM 2
WHEN THE NOISE IS GAUSSIAN

$Y = f(Z)$	$\Omega(Y)$	Conditions
$Y = h + Z$	$Q\left(\frac{Y}{\sigma}\right)$	$\frac{h-D}{\sqrt{2}\sigma} > t_n, n \text{ odd}$
$Y = (h + Z)^2$	$Q\left(\frac{\sqrt{Y}}{\sigma}\right)$	$h-D > 0, n \text{ odd}$

III. APPLICATION TO THE GAUSSIAN CASE

In this section we shall examine closely the implications of the technical conditions (2.9)–(2.11). Since their validity is dependent on the choice of the function $\Omega(\cdot)$, we shall consider the two special cases considered in Section I, namely, (1.7) and (1.8), corresponding to $\Omega(x) = Q(x/\sigma)$ and $\Omega(x) = Q(\sqrt{x}/\sigma)$. For other possible choices of $\Omega(\cdot)$ (see [1]) the analysis can be worked out by straightforward extension of the techniques presented here.

Consider first condition (2.9), which requires $\min Y > 0$. This is equivalent to the requirement that $h - D > 0$ (the open-eye assumption) when (1.7) holds. If (1.8) was chosen, condition (2.9) is always satisfied, but the eye must be open for (1.8) to hold. Thus, $h - D > 0$ is a necessary condition in both cases.

Consider then (2.10). If $\Omega(x) = Q(x/\sigma)$, we have for $k \geq 1$:

$$(-1)^{n+1-k} \Omega^{(k)}(a) = \frac{(-1)^{n+1}}{2^{k/2} \sigma^k \sqrt{\pi}} e^{-a^2/2\sigma^2} H_{k-1}\left(\frac{a}{\sqrt{2}\sigma}\right), \quad (3.1)$$

where $H_n(\cdot)$ is the Hermite polynomial of degree n [3, p. 691]. Hence, for (2.10) to be satisfied, the following must hold:

$$(-1)^{n+1-k} H_{k-1}\left(\frac{a}{\sqrt{2}\sigma}\right) \geq 0, \quad k = 1, 2, \dots, n, \quad (3.2)$$

and

$$(-1)^{n+1} Q\left(\frac{a}{\sigma}\right) \geq 0. \quad (3.3)$$

If n is even, (3.3) cannot hold true; if n is odd, (3.3) is always satisfied, and a sufficient condition for (3.2) to hold is that $a/\sqrt{2}\sigma$ be larger than the largest root, say t_{n-1} , of $H_{n-1}(x)$, $k = 1, 2, \dots, n$. Moreover, since the value of t_{n-1} increases with increasing k , the sufficient condition becomes

$$\frac{a}{\sqrt{2}\sigma} \geq t_{n-1}, \quad (3.4)$$

where t_{n-1} denotes the largest root of $H_{n-1}(x)$, and $a = h - D$.

If on the contrary $\Omega(x) = Q(\sqrt{x}/\sigma)$, from [1] we have

$$(-1)^{n+1-k} \Omega^{(k)}(a) = (-1)^{n+1-k} g_k(a) e^{-a^2/2\sigma^2}, \quad k = 1, \dots, n, \quad (3.5)$$

where $g_k(x)$ is a nonzero polynomial with nonnegative coefficients. Eq. (3.5) shows that, since $a > 0$, for n odd (2.10) is always satisfied.

Consider finally (2.11). The computations just performed can be used to show that for $\Omega(x) = Q(x/\sigma)$ (2.11) holds true for n odd provided that

$$\frac{h-D}{\sqrt{2}\sigma} > t_n \quad (3.6)$$

Since $t_n > t_{n-1}$, this condition implies (3.4) as well. Similarly, for $\Omega(x) = Q(\sqrt{x}/\sigma)$ and n odd it can be seen that (2.11) always holds true.

The conditions for the validity of Theorem 2 are summarized in Table I. It is seen that in both cases it is necessary that n be

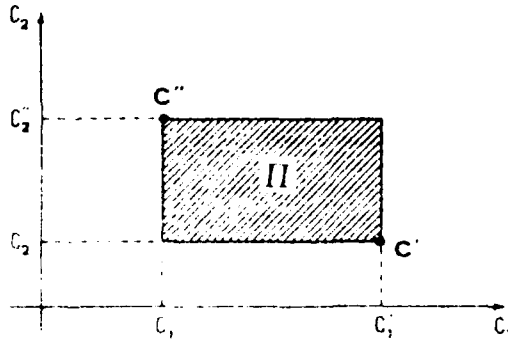


Fig. 2. Upper and lower bounds to error probability for a binary communication system. Disturbances are additive Gaussian noise and intersymbol interference generated by a Gaussian pulse whose samples can fluctuate in intervals of relative width ϵ percent around their nominal values.

odd. However, this is not a restriction. In fact, it can be easily proved that it is sufficient to change $\Omega(\cdot)$ into $-\Omega(\cdot)$ (i.e., to look for bounds to $-E(\cdot)$) to make the theorem hold for n even as well.

IV. NUMERICAL RESULTS

In this section we present some numerical results for the error bound theory discussed in Sections II and III. In particular, we shall consider a binary digital system perturbed by additive Gaussian noise and intersymbol interference.

$$Z = \sum_{i=1}^L \alpha_i h_i, \quad (4.1)$$

where α_i , $i = 1, \dots, L$, are independent identically distributed random variables taking on values ± 1 with equal probabilities. We assume that the samples h_i , $i = 1, \dots, L$, of the channel impulse response are inaccurately known, so that each of them lies in a finite interval whose boundaries are known:

$$h_i \leq h_i \leq h_i, \quad i = 1, \dots, L. \quad (4.2)$$

Consider the moments of Z ; all of its odd-order moments are zero, whereas the even-order moments can be given the form

$$E[Z^{2m}] = \sum (2m_1, 2m_2, \dots, 2m_L) h_1^{2m_1} \dots h_L^{2m_L}, \quad (4.3)$$

where the summation is extended over all the possible L -tuples of different indices m_1, \dots, m_L such that $2m_1 + \dots + 2m_L = 2m$.

The notation $(2m_1, 2m_2, \dots, 2m_L)$ denotes the multinomial coefficient, which is positive, and we have taken into account the fact that $E[\alpha_i^{2m}] = 1$ for all m . Eq. (4.3) shows that the even-order moments of Z are polynomials in the variates h_1^2, \dots, h_L^2 with nonnegative coefficients, which implies that they are increasing functions of h_1^2, \dots, h_L^2 . This shows that, by substituting for h_1^2, \dots, h_L^2 in (4.3) their minimum and maximum values, respectively, we get two sequences of moments $\{\mu_i\}_{i=1}^n$ and $\{\bar{\mu}_i\}_{i=1}^n$ such that

$$\mu_i \leq E[Z^i] \leq \bar{\mu}_i \quad (4.4)$$

Consider now the random variables $X = h + Z$ and $Y = (h + Z)^2$; the moments of Y can be expressed in both cases through the moments of Z as follows:

$$E[Y^p] = E[(h + Z)^{2p}] = \sum_{i=0}^{2p} \binom{2p}{i} h^i \mu_{2p-i}, \quad (4.5)$$

where $p = 1, 2$. Since $E[Y^p]$ is expressible as a linear combination of moments μ_i with positive coefficients, substitution of $\{\mu_i\}_{i=1}^n$

and $\{\bar{\mu}_i\}_{i=1}^n$ for $\{\mu_i\}_{i=1}^n$ in (4.5) will provide two sets of moments $\{c_i\}_{i=1}^n$ and $\{\bar{c}_i\}_{i=1}^n$ that satisfy (1.10), and hence allow one to use the theory outlined in Section II.

These results have been applied to a system whose nominal impulse response is specified by the Gaussian pulse $\exp - (8t/5T)^2$ sampled at multiples of T . Then $h = 1$, L is taken as 6, and

$$D = \sum_{i=1}^6 |h_i| = 0.15468$$

We assume that the h_i , $i = 1, \dots, 6$, can fluctuate around their nominal values within an interval of width Δh_i . We define

$$\epsilon \triangleq \frac{\Delta h_i}{h_i} \cdot 100\%$$

and the signal-to-noise ratio (SNR)

$$\text{SNR} \triangleq 20 \log \frac{h}{\sigma}$$

By taking $n = 10$ moments, and choosing $Y = h + Z$, we obtain the numerical results shown in Fig. 2 for the error probability of the digital system.

V. CONCLUSION

The theory of moment bounds was extended to encompass a situation in which the moments of the random variable representing the interference are inaccurately known. In this case, the evaluation of upper and lower bounds to the error probability is reduced to the computation of "standard" moment bounds, in which the sequence of moments is constructed on the basis of the upper and lower bounds to the moments of Z .

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Appendix C

OPTIMAL LINEAR RECEIVING FILTER FOR DIGITAL TRANSMISSION OVER NONLINEAR CHANNELS

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ABSTRACT

The optimum linear receiving filter for digital transmission over a nonlinear channel is specified. Under the assumption that the channel has a finite memory and that the noise is additive Gaussian, an optimum receiver is sought in the form of a linear receiving filter followed by a symbol-rate sampler and a memoryless decision device. The receiving filter is chosen so as to minimize the mean-square error between the input to the decision device and the transmitted symbol.

It is shown that the structure of this optimum receiving filter corresponds to a bank of matched filters, each followed by a linear transversal filter. The number of matched filters is equal to the number of linearly independent waveforms that can be observed at the channel output in a symbol period. This result is a generalization of a fact which is well-known for linearly modulated signals transmitted over linear channels. Also, the performance of the receiver so optimized is evaluated and compared to conventional designs.

STATEMENT OF THE PROBLEM AND MOTIVATION OF THE WORK

Let $\{q(t; i)\}_{i=1}^M$ be a set of finite-energy waveforms with common duration T , and $\{\xi_n\}_{n=-\infty}^{\infty}$ a wide-sense stationary sequence of random variables taking values in the set $\{1, \dots, M\}$. Consider then the signals

$$x(t) = \sum_{n=-\infty}^{\infty} q(t - nT; \xi_n) \quad (1.1)$$

and

$$y(t) = x(t) + v(t) \quad (1.2)$$

where $v(t)$ is a white noise process independent of the sequence $\{\xi_n\}$, with two-sided power spectral density $N/2$. The problem we consider in this paper is the following: for a given deterministic function $u(\cdot)$ of the random variables ξ_n , and a given time $t_n = t_0 + nT$,

$0 \leq t_n < T$, choose the impulse response $u(t)$ of a linear time-invariant system such that the quantity

$$\mathcal{E} = E[|y(t_n) * u(t_n) - a(\xi_n)|^2] \quad (1.3)$$

(where $*$ denotes convolution) is a minimum.

The problem just stated is the formulation, in a rather general form, of a problem arising in data transmission. We want to specify an optimum receiver in the form of a linear receiving filter followed by a symbol-rate sampler and a memoryless decision device. The channel is assumed to be time-invariant (linear or nonlinear), and to have a finite memory. The data symbols may be encoded, and the memory introduced by the encoder is assumed to be included in the channel part of the data transmission model. The channel consists of a noiseless part followed by the addition of noise. This noiseless part is depicted in Fig. 1. A sequence $\{a_n\}$ of independent data symbols is fed into a shift register with $L-1$ delay elements (denoted by D in the figure). The L -tuple $(a_n, a_{n-1}, \dots, a_{n-L+1})$ denotes the state of the channel at the discrete time n . We assume that there are M different such L -tuples, that we put in a one-to-one correspondence with the integers $1, \dots, M$. In any time interval of duration T there is a finite number of possible signals at the output of the noiseless part of the channel, and they depend on the state of the channel. We denote by $q(t - nT; \xi_n)$ the waveform at the channel output in the interval $nT \leq t < (n+1)T$. Thus, before the addition of noise the transmitted signal has the form (1.1) and the signal observed at the receiver's front end has the form (1.2). For a receiver structure as depicted in Fig. 2, we want to choose $u(t)$ such that the sample $y(t)$ taken at time $t = t_n$ is as close as possible to the transmitted symbol a_n . If we define $a(\xi_n) = a_n$, the quantity

$$y(t_n) * u(t_n) - a(\xi_n) \quad \text{in (1.3) is the difference}$$

between the input to the decision device and the corresponding transmitted symbol. Thus minimization of \mathcal{E} provides a minimum square error receiver.

Optimization of the receiver under this minimum mean-square error criterion for a linear channel and pulse-amplitude modulation received much attention in the literature (see, e.g., [1]). More recently, optimum receiving filters were considered for nonlinear channels as well [2,3].

That work was motivated by digital satellite communication, in which amplifiers are used that operate at or near saturation for better efficiency. Fredricson [2] considered a quadrature PSK system, and used an approximate model for the nonlinear amplifier. Mesiya, McLane and Campbell [3] specified an optimum receiving filter for binary PSK.

This paper is an extension of the previous work done in this area. Our approach is general enough to encompass a number of multilevel modulation schemes and any kind of noiseless nonlinear channels. Consideration of coded symbols is also possible, as well as the addition of interfering signals other than white noise. However, for sake of simplicity we shall not deal with these more general situations here. In the following, we shall provide a combined analytical and simulation technique to specify the optimum receiver filter. From a theoretical point of view, it is interesting to observe that this optimum filter can be thought of as composed of a bank of filters, matched to a set of $n < M$ waveforms that form a basis for $\{q(t;i)\}_{i=1}^M$, in

cascade with n infinite-length transversal filters. This is a generalization of a result which is well-known in the case of PAM modulation and linear channels [1].

THE GENERAL SOLUTION

To formulate in the frequency domain the problem of minimizing \mathcal{E} in (1.3), let us denote with $U(f)$ the transfer function of the receiving filter, and with $X(f)$ the Fourier transform of the signal $x(t)$ defined in (1.1). We have

$$X(f) = \sum_{n=-\infty}^{\infty} Q(f; \xi_n) \exp(-j2\pi f n T) \quad (2.1)$$

where $\{Q(f; i)\}_{i=1}^M$ are the Fourier transforms of the waveforms $\{q(t; i)\}_{i=1}^M$. Eq. (1.3) can be rewritten as follows:

$$\mathcal{E} = \sigma_a^2 + \iint_{-\infty}^{\infty} U(f) \Gamma(f, f') U^*(f') \exp[j2\pi(f-f')t_n] df df' - 2\Re \int_{-\infty}^{\infty} V(f) U(f) df + N_0/2 \int_{-\infty}^{\infty} |U(f)|^2 df \quad (2.2)$$

where \Re denotes real part, the superscript * denotes complex conjugate and the following definitions have been used:

$$\Gamma(f, f') = E[X(f)X^*(f')] \quad (2.3)$$

$$V(f) = \exp(-j2\pi f t_n) E[n(\xi_n)X^*(f)] \quad (2.4)$$

and

$$\sigma_a^2 = E[|u(\xi_n)|^2] \quad (2.5)$$

Using standard variational calculus techniques, it can be shown that a necessary and sufficient condition for $U(f)$ to minimize \mathcal{E} is that it be solution of the following integral equation:

$$\int_{-\infty}^{\infty} \Gamma^*(f, f') U(f') \exp(-j2\pi(f-f')t_n) df' + N_0/2 U(f) = V(f) \quad (2.6)$$

Thus far, we have not yet exploited our knowledge of the structure of $X(f)$, provided by (2.1). Recalling that the sequence (ξ_n) is stationary, we can define

$$R(f, f'; n-m) = E[Q(f; \xi_n) Q^*(f'; \xi_m)] \quad (2.7)$$

so that $\Gamma(f, f')$ can be written in the form:

$$\Gamma(f, f') = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} R(f, f'; n-m) \exp(-j2\pi(nf-mf')T) = 1/T \sum_{n=-\infty}^{\infty} \left[\sum_{l=-\infty}^{\infty} R(f, f'; l) \exp(-j2\pi lfT) \right] \delta(f-f'-n/T) \quad (2.8)$$

where the equality

$$\sum_{n=-\infty}^{\infty} \exp(-j2\pi n(f-f')T) = 1/T \sum_{n=-\infty}^{\infty} \delta(f-f'-n/T) \quad (2.9)$$

has been used. If we define the functions

$$A(f, f') = \sum_{l=-\infty}^{\infty} R(f, f'; l) \exp(-j2\pi lfT) \quad (2.10)$$

and

$$G_n(f) = A(f, f-n/T) \quad (2.11)$$

$\Gamma(f, f')$ can be rewritten as

$$\Gamma(f, f') = 1/T \sum_{n=-\infty}^{\infty} G_n(f) \delta(f-f'-n/T) \quad (2.12)$$

so that the integral equation (2.6) takes the form:

$$1/T \sum_{k=-\infty}^{\infty} G_k^*(f) U(f-k/T) \exp(-j2\pi k t_n/T) + N_0/2 U(f) = V(f) \quad (2.13)$$

The theorem that follows provides, under certain conditions, a closed-form solution of (2.13).

Theorem. If the functions $V(f)$ and $G_n(f)$ in (2.13) can be written in the form:

$$V(f) = Q^\dagger(f) B(f) \quad (2.14)$$

and

$$G_n(f) = Q^\dagger(f) \mathcal{P}(f) Q^*(f-n/T) \exp(-j2\pi n t_n/T) \quad (2.15)$$

where the superscript T denotes transpose, a dagger denotes conjugate transpose, $Q(f)$ is a column M-vector whose components are the Fourier transforms of the waveforms $q(t; i)$, $1 \leq i \leq M$, $B(f)$ is a column M-vector of frequency functions periodic with period $1/T$, and $\mathcal{P}(f)$ is a MxM matrix of frequency functions periodic with period $1/T$, then (2.13) admits the solution

$$U(f) = \int_{-\infty}^{\infty} u(t) e^{-j2\pi ft} dt \quad (2.16)$$

where $\Gamma(f)$ is a column M -vector of frequency components periodic with period $1/T$.

Substitution of (2.14) - (2.16) into (2.13) gives:

$$U(f) = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} q(t-k/T) Q^*(f-k/T) e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} \Gamma(f) e^{-j2\pi ft} dt \quad (2.17)$$

where Γ denotes the $M \times M$ identity matrix. Equivalently, (2.17) can be rewritten in the form

$$Q^*(f) C(f) = U(f) \quad (2.18)$$

where $C(f)$ is the M -vector, periodic with period $1/T$, defined by

$$C(f) = H(f) \Gamma(f) = H(f) \quad (2.19)$$

and

$$H(f) = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} q(t-k/T) Q^*(f-k/T) e^{-j2\pi ft} dt \quad (2.20)$$

Equation (2.19) can be rewritten in explicit form in the time domain. If $c_i(t)$, $1 \leq i \leq M$, denote the inverse Fourier transforms of the components of the vector $C(f)$, the periodicity of $C(f)$ implies that $c_i(t)$ have the following form:

$$c_i(t) = \sum_{j=-\infty}^{\infty} \delta(t-jT) \quad (2.21)$$

hence, the time-domain version of (2.18) is

$$\sum_{j=-\infty}^{\infty} \sum_{i=1}^M q_i(t-jT) Q^*(f-k/T) = 0 \quad (2.22)$$

Thus, if the waveforms $q(t;i)$ are linearly independent, eq. (2.22), and hence (2.18), can hold if and only if all the Q_{ij} are equal to zero. This entails $C(f) = 0$, i.e., assuming that $H(f)$ is nonsingular.

$$\Gamma(f) = H^{-1}(f) B(f) \quad (2.23)$$

Eq. (2.23) shows that $\Gamma(f)$ is periodic with period $1/T$, as it should be.

Assume now that the waveforms in $\{q(t;i)\}_{i=1}^M$

are not linearly independent. In this situation, let $\{\psi(t;i)\}_{i=1}^N$, $N < M$, denote a set of linearly

independent basis functions for $\{q(t;i)\}_{i=1}^M$. If

$\{\psi(f;i)\}_{i=1}^N$ denotes the set of Fourier transforms of these basis functions, and $\Psi(f)$ is the N -vector whose components are these Fourier transforms, an n, M matrix D can be found such that

$$Q(f) = \Psi(f) D \quad (2.24)$$

In this case, (2.18) becomes

$$\Psi^*(f) D C(f) = U(f) \quad (2.25)$$

whose solution is now given by

$$D C(f) = 0 \quad (2.26)$$

or equivalently,

$$\Gamma(f) = H^{-1}(f) D^* D B(f) \quad (2.27)$$

where D^* is the $M \times n$ matrix called the Moore-Penrose pseudoinverse of D . The theorem is proved.

This theorem shows that the optimum receiving filter in our situation can be thought of as composed of a bank of filters with transfer functions $Q^*(f;i)$, each cascaded to an infinite-length transversal filter. The filters $Q^*(f;i)$ are matched to the waveforms observed at the output of the memoryless part of the channel.

When the waveforms $\{q(t;i)\}_{i=1}^M$ are not linearly independent, we can substitute the bank of M filters $\{Q^*(f;i)\}_{i=1}^M$ with n filters, matched to the basis functions $\{\psi_i(f)\}_{i=1}^N$, according to the equation

$$U(f) = \Psi^*(f) D C(f) \quad (2.28)$$

COMPUTATION OF THE TRANSFER FUNCTION OF THE OPTIMUM FILTER

From the results presented in the previous section, it is seen that the transfer function $U(f)$ of the optimum receiving filter has the general expression

$$U(f) = Q^*(f) H^{-1}(f) D^* D B(f) \quad (3.1)$$

where $B(f)$ is obtained through (2.14), and $H(f)$ is defined by (2.20) in conjunction with (2.15). This is valid, however, only if the conditions (2.14) and (2.15) hold. In this section we shall show that, with the model assumed for the channel (see Fig. 1) the hypotheses of the theorem hold true, and an algorithm will be provided to compute the quantities involved in (3.1).

To do this, we observe that under our assumptions the discrete sequence $(\xi_n)_{n=-\infty}^{\infty}$ forms an M -state fully regular homogeneous Markov chain. Let P denote the transition probability matrix of this chain, i.e.,

$$(P)_{ij} = P\{\xi_{n+1} = j | \xi_n = i\} \quad (3.2)$$

and define

$$L(f) = \sum_{n=1}^{\infty} P^n \exp(-j2\pi n f T) \quad (3.3)$$

and

$$A = \{a(1), a(2), \dots, a(N)\}^T \quad (3.4)$$

Then, it can be shown that the expression for the optimum filter is

$$U(f) = \exp(-j2\pi f t_0) Q^H(f) H^{-1}(f) D^H D \mathcal{F}^H(f) A \quad (3.5)$$

where $H(f)$ is given by (2.20), $\mathcal{F}(f)$ by (2.36) - (2.41), and $Q(f)$ is the M -vector whose components are $Q(f;1), \dots, Q(f;M)$.

As an example of application, consider binary PSK transmission over a nonlinear channel consisting of a 4th-order Butterworth filter with 3-dB bandwidth 1.08/T cascaded to a nonlinear amplifier (a TWT exhibiting both AM/AM and AM/PM conversion); Fig.4 shows the transfer function of the optimum filter for a channel with the nonlinearity removed. The power spectra of the received signal are also shown for comparison. The performance of the optimum filter is shown in Fig.5. For comparison's sake, the mean-square error resulting from a 2-pole Butterworth receiving filter with optimum 3-dB bandwidth is also shown.

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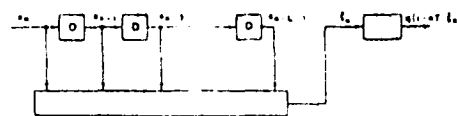


Fig.1 - Model of a discrete-input, continuous-output finite-memory channel

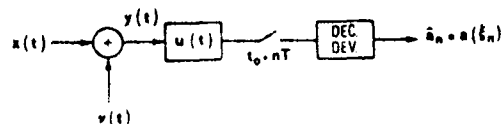


Fig.2 - Model of the receiver

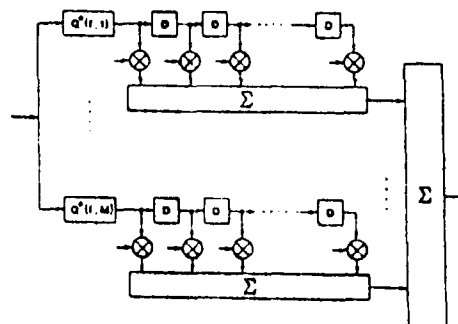


Fig.3 - Structure of the optimum receiving filter

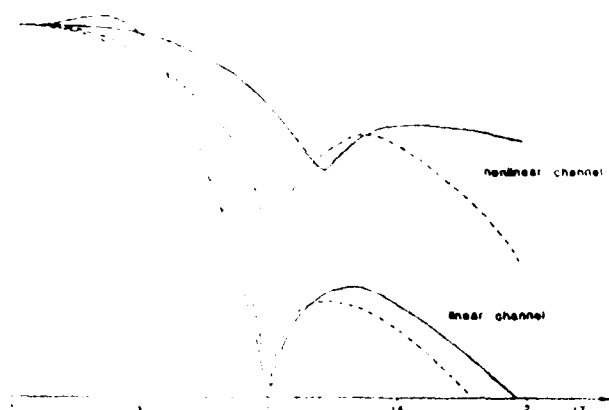


Fig. 4 - — transfer function of the optimum receiving filter
 --- power spectral density of the received signal
 20-PSK, 4th-order Butterworth TX filter
 $2B_bT = 1.00$
 $\gamma = 1.00$

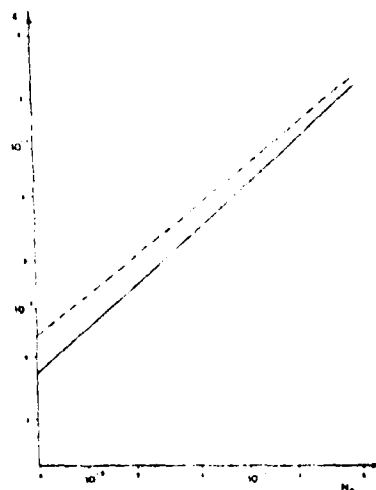


Fig. 5 - 20-PSK; 4th-order Butterworth TX filter with $2B_bT = 1.00$. Mean-square error with optimum filter (continuous line) and with 2-pole Butterworth filter (dashed line).

Appendix D

MENSIONAL MODULATION AND CODING FOR DIGITAL TRANSMISSION

and M. Ella (*)

performance in digital data transmission can be achieved, without increase neither in bandwidth nor power, by adding a certain amount of complexity in the transmission system. It is to do that are an increase in dimensionality of the signal constellation used for transmission, or the use of multilevel trellis codes ("Ungerboeck") codes. We consider here a combination of both techniques. A class of multidimensional constellations, called "generalized codes" is introduced, and it is shown an Ungerboeck code can be designed using one of those as an

dimensions. Conventional quadrature amplitude modulation (QAM) and phase-shift keying (PSK) use the in-phase and quadrature components of a carrier to build up a two-dimensional (2D) signal space. Now, by choosing an appropriate alphabet, the performance can be improved by going to higher-dimensional signal spaces.

Although our main concern here is the discussion of specific designs, it appears also natural to investigate what performance improvement can theoretically be achieved when this dimensionality is increased. Using sphere-packing arguments, it was shown in [10] that for a signalling alphabet whose elements have the same energy E , the dimensionality n , the number of signals M , and the minimum Euclidean distance d_{\min} in the signal space are related by the bound

$$M < 2 / I_q((n-1)/2, 1/2)$$

where $q = d_{\min}^2 / 4E$, and $I_q(x, y)$ is the incomplete beta-function.

INTRODUCTION

Digital communication over radio channels, both available spectrum and transmitted power are generally limited. Thus, to cope with the increasing demand for digital communication services, more efficient modulation techniques are called for, the search for bandwidth and efficient modulation systems has become a very active research area. In this field it has been widely accepted that the trade-off factor is relevant in the relationship between bandwidth and power, and, as a consequence, the complexity of the modulation system. In other words, the additional amount of complexity introduced at the transmitting and the receiving ends of the system, its performance can be improved without increasing neither bandwidth nor power.

A possible way to allocate the available complexity is to use larger alphabets in higher

Similar results can be obtained by allowing the signals to have more than one energy level. To do this, we assume that the energy spacing between adjacent levels is equal to the minimum squared distance between signals, and apply the bound just given to each energy level. This results in a set of nonlinear equations (with as many equations as energy levels), which can be solved numerically and whose results are shown in Fig. 0. In this figure, the abscissa $\log_2 M/n$ represents the information rate in bits per dimension, and the ordinate is labeled by a parameter related to the error performance of the signal set when used on an additive white Gaussian noise channel. More specifically, the error probability turns out to be a monotone decreasing function of the product of E_b/N_0 (the ratio between the average energy per bit and the noise power spectral density) and $\log_2 M$, where $\delta_{\min}^2 = d_{\min}^2 / E$. Thus, given two systems operating at the same average energy E and information rate, the one having a larger value of $\log_2 M$ will exhibit

a smaller error probability, at least for high E_b/N_0 . The performance chart of Fig. 9 was first proposed in [11].

From Fig. 9, the gain involved in an increase of signal dimensionality is apparent. Also, it is seen that as the information rate gets larger, increasing the number of energy levels provides a substantial improvement in performance only for small values of n .

In practice, 4D signal spaces can be obtained simultaneously using two phase-orthogonal carriers with two orthogonally polarized electromagnetic waves [1,2] or with two time-division [12] or frequency-division multiplexed signals. Results on specific designs of 4D and 16 signal sets can be found in [1-8]. They show for example that, as compared to 2D QAM, the signal power that can be saved to achieve a bit error rate of 10^{-5} is about 1.2 dB (when $n=4$) and 2.1 dB (when $n=8$).

Another technique was recently proposed by Ungerboeck, and has been receiving considerable attention. It is based on a combined modulation-coding approach [6-8]. In standard applications of error-control codes, extra bits must be added to the transmitted signal sequence, with the transmitter operating at a higher rate, and hence requiring a larger bandwidth. The overhead rate of coding can reduce the power requirement necessary to achieve a given performance level, so that this gain in performance is usually referred to as the coding gain, and compensate for the additional penalty. In other words, the increase in increased complexity and decreased bandwidth efficiency for increased power efficiency. With trellis codes, one can avoid the loss in bandwidth efficiency because the redundancy required by the coding process is provided by increasing the number of states instead of the number of code words. Codes can be decoded using a Viterbi algorithm, and a considerable gain can be obtained at the expense of a reasonable complexity.

In this paper, we consider a combination of multidimensional alphabets and Ungerboeck codes. An idea which was first suggested in [1-2]. In particular, we consider two classes of alphabets, namely "group codes" (as defined by Ungerboeck [6]), which are subject to a power constraint, and "generalized group codes", which we introduce here, which are based on a peak-to-average constraint. These alphabets exhibit a high degree of symmetry, a property which is important in order to obtain good performance with Ungerboeck

codes. Generalized group codes form an exceedingly large class of alphabets, and in fact most of the alphabets that have been proposed so far for multidimensional signalling belong to this family. Techniques are shown that allow multidimensional-alphabet Ungerboeck codes to be designed: they are based on a partition of the alphabet stemming from the partition of a group into cosets of a convenient subgroup.

2.0 GENERALIZED GROUP CODES FOR THE GAUSSIAN CHANNEL

Consider a set of K n -vectors $\{x_1^{(0)}, \dots, x_K^{(0)}\}$, and a set of $n \times n$ orthogonal matrices S_1, \dots, S_L that form a finite group G under matrix multiplication. The set of vectors

$$x_j^{(i)} = S_i x_j^{(0)} \quad i=1, \dots, L, \quad j=1, \dots, K.$$

is called a "generalized group code" (GGC) if the action of the matrix group G on the "initial set" of vectors is such that each of them is transformed into an equal number of distinct vectors. The special case $K=1$ has been studied in [9], and gives rise to a class of alphabets called "group codes for the Gaussian channel". Notice that the number of distinct vectors in a GGC may be less than KL (but is always a multiple of K), and these vectors may not span the Euclidean n -dimensional space.

If $|x|$ denotes the Euclidean length of a vector x , the quantity $|x|^2$ is proportional to the energy of the signal associated with x .

As orthogonal matrices transform a vector into another having the same length, a GGC has as many energy levels as there are in the "initial set" $\{x_1^{(0)}, \dots, x_K^{(0)}\}$. In the special case $K=1$, all the code vectors have the same length.

Let now $R_j^{(i)}$ denote the maximum likelihood (ML) region associated with the code vector $x_j^{(i)}$, i.e., the set of points in the Euclidean n -space at least as close to $x_j^{(i)}$ as to any other code vector. Then, for a GGC every ML region is congruent to one among $R_1^{(0)}, \dots, R_K^{(0)}$. In other words, the ML regions can have no more than K different shapes. Consequently, if a ML receiver is used and $P(e|x_j^{(i)})$ denotes the error probability when $x_j^{(i)}$ is transmitted, this can take no more than

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seen that a sufficient condition is $C_2 = 0$, or $C_1 = 0$. The first condition is equivalent to state that for all the encoder states the symbols available to the encoder have zero mean. For equally likely source symbols, this condition reduces to the requirement that all the columns of the table displaying the values of the function $h(\cdot, \cdot)$ have a zero arithmetic mean. The second condition is equivalent to state that, for each encoder state, the average of the symbols forcing the encoder to that state have zero mean.

Consider then the line spectrum (2.19). A sufficient condition for it to be zero is that $\mu = 0$, i.e., the average of the symbols at the encoder output be zero. From (2.14) we see that if $C_2 = 0$ then $\mu = 0$, so that if $C_2 = 0$ encoding does not alter at all the power spectrum of the line signal.

Let us now restrict our attention to the Ungerboeck code described in

1. It is easily seen that if the source symbols are equally likely we have for this code $\tilde{C}_2 = 0$. More generally, the condition $\tilde{C}_2 = 0$ is satisfied by all Ungerboeck codes designed so far, at least when the source symbols are equally likely. This is due to the highly symmetrical structure exhibited by good Ungerboeck codes.

ally, the fourth quantity of interest is the mean-square value of the encoded

$$\begin{aligned} C_0 &= \sum_{k=0}^{L-1} \sum_{\ell=1}^N q_k p_\ell |h(k, S_\ell)|^2 \\ &= \sum_{k=0}^{L-1} q_k \mathbf{U}_k^* \mathbf{D} \mathbf{U}_k \end{aligned} \quad (2.15)$$

the asterisk denotes conjugate.

are now in a position to express the power spectrum of the modulated signal

(1.3). We have

$$\mathcal{G}_x(f) = \mathcal{G}_x^{(c)}(f) + \mathcal{G}_x^{(d)} \quad (2.16)$$

$\mathcal{G}_x^{(c)}(f)$, the continuous part, is given by

$$\mathcal{G}_x^{(c)}(f) = |Q(f)|^2 \left\{ \frac{1}{T} (C_0 - |\mu|^2) + \frac{2}{T} \Re [\mathbf{C}_1^* (\mathbf{I} - \mathbf{P}^\infty) \mathbf{B}(f) \mathbf{C}_2'] \right\} \quad (2.17)$$

$Q(f)$ is the Fourier transform of the waveform $q(t)$ available to the modulator, \Re denotes real part, \mathbf{I} denotes the $N \times N$ identity matrix, and

$$\begin{aligned} \mathbf{B}(f) &= e^{-j2\pi f T} \sum_{n=0}^{\infty} (\mathbf{P} - \mathbf{P}^\infty)^n e^{-j2\pi f n T} \\ &= [e^{j2\pi f T} \mathbf{I} - (\mathbf{P} - \mathbf{P}^\infty)]^{-1} \end{aligned} \quad (2.18)$$

for more details).

The discrete part of the power spectrum (line spectrum) is given by

$$\mathcal{G}_x^{(d)}(f) = |\mu|^2 \frac{|Q(f)|^2}{T^2} \sum_{n=-\infty}^{\infty} \delta(f - n/T). \quad (2.19)$$

conclusions

are now in a position to state simple sufficient conditions that a code must

for its spectrum to be equal to a scalar multiple of $|Q(f)|^2$. From (2.18) it can be

exists under our assumptions, and has all its rows equal to the probability vector p .

Let us now evaluate four quantities which play an important role in the discussions that follow. The first among them is the average value of U_k taken over the source symbols, i.e.,

$$C_2 = \sum_{k=0}^{L-1} q_k U_k. \quad (2.10)$$

The j -th component of C_2 turns out to be the average of the symbols available to the encoder when it is in state S_j .

The second quantity to be defined is the N -vector C_1 whose j -th component is the average of the coded symbols that, when output by the encoder, force it to the state S_j . This j -th component of C_1 is then given by (see [3] for details):

$$[C_1]_j = \sum_{k=0}^{L-1} \sum_{\ell=1}^N q_k p_\ell [E_k]_{\ell,j} h(k, S_\ell) \quad (2.11)$$

(recall that $[E_k]_{\ell,j} = 1$ only if the source symbol k takes the encoder from state S_ℓ to state S_j). If we define the $N \times N$ diagonal matrix

$$D = \text{diag} (p_1 \ p_2 \ \dots \ p_N) \quad (2.12)$$

we have from (2.11)

$$C_1 = \sum_{k=0}^{L-1} q_k U_k D E_k. \quad (2.13)$$

Our third quantity is the average symbol at the output of the encoder. We have

$$\begin{aligned} \mu &= \sum_{k=0}^{L-1} \sum_{\ell=1}^N q_k p_\ell h(k, S_\ell) \\ &= \sum_{k=0}^{L-1} q_k p U'_k \\ &= p C'_2 \end{aligned} \quad (2.14)$$

$$U_3 = (w^6 \ w^7 \ w^4 \ w^5) \quad (2.2d)$$

$$w = e^{j\pi/4}, \quad (2.3)$$

denoting by q_k , $k = 0, 1, \dots, L-1$, the probabilities of the source symbols, i.e.,

$$q_k = P\{a_n = k\}, \quad k = 0, 1, \dots, L-1, \quad (2.4)$$

state sequence (σ_n) is a homogeneous Markov chain with transition probability

$$P = \sum_{k=0}^{L-1} q_k E_k \quad (2.5)$$

e

$$[P]_{ij} = P\{\sigma_{n+1} = S_j | \sigma_n = S_i\} \quad (2.6)$$

[3] for a proof). We assume that the chain is fully regular (see [4] for the relevance of this assumption in the computation of power spectra). The matrix P provides all the information we need about the statistics of the sequence of the encoder states. In particular, the stationary state probabilities

$$p_i = P\{\sigma_n = S_i\}, \quad i = 1, 2, \dots, N \quad (2.7)$$

are the elements of the row N -vector p obtained as the solution of the equation

$$p = p P \quad (2.8)$$

subject to the condition $\sum_{i=1}^N p_i = 1$. The limiting transition probability matrix

$$P^\infty = \lim_{n \rightarrow \infty} P^n \quad (2.9)$$

For our computations, the following quantities are to be defined:

The *state transition matrices* E_k , $k = 0, 1, \dots, L-1$, which are the $N \times N$ matrices whose entry $[E_k]_{i,j}$ is equal to 1 if $g(k, S_i) = S_j$, and zero elsewhere.

The row vectors U_k , $k = 0, 1, \dots, L-1$, whose N entries are the coded symbols according to the rule $[U_k]_i = h(k, S_i)$.

In words, the matrix E_k has a 1 in row i and column j if the source symbol k forces a transition of the encoder from state S_i to state S_j , and a zero elsewhere. The vector U_k includes the coded symbols corresponding to the source symbol k , for different states of the encoder. For the example of Fig.1, we have

$$E_0 = E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (2.1a)$$

$$E_2 = E_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.1b)$$

and

$$U_0 = (w^0 \ w^1 \ w^2 \ w^3) \quad (2.2a)$$

$$U_1 = (w^4 \ w^5 \ w^6 \ w^7) \quad (2.2b)$$

$$U_2 = (w^2 \ w^3 \ w^0 \ w^1) \quad (2.2c)$$

the modulator, the modulated signal can be written in the form

$$x(t) = \sum_{n=-\infty}^{\infty} u_n q(t - nT) \quad (1.3)$$

We want to compare the power spectrum $\mathcal{G}_x(f)$ of the signal $x(t)$ with the power spectrum $\mathcal{G}_s(f)$ of the signal $s(t)$ which would result if the source symbols were not encoded, but sent directly to a ^{linear} modulator. Simple sufficient conditions will be found showing that a properly designed Ungerboeck code does not alter the power spectrum of the line signal. This provides a formal proof of the often-claimed fact that Ungerboeck codes "do not expand the bandwidth".

1. The power spectrum of the coded signal

In this section we shall present the power spectrum of the signal $x(t)$ defined in (1.3). As the computation can be done by using results obtained by Cariolaro *et al.* [3], we shall not repeat their derivation here, and focus instead on the meaning of the quantities involved.

To describe the encoder operation, we need to specify the functions $h(\cdot, \cdot)$ and $g(\cdot, \cdot)$ of (1.1) and (1.2). This will be given in tabular form, by providing two tables whose rows are labeled according to the values taken on by the source symbols a_n (say, $0, 1, \dots, L-1$) and whose columns are labeled according to the values taken on by the states σ_n (say, S_1, S_2, \dots, S_N). In the first table we display the values of $h(a_n, \sigma_n)$, and in the second the values of $g(a_n, \sigma_n)$. As an example, Fig.1 provides the two tables needed to describe the 4-state Ungerboeck code of [1, Fig.7].

1. Statement of the problem and motivation of the work

In this paper we consider the power spectrum of the digital signal resulting after channel coding and multilevel/phase modulation. The family of codes considered are those introduced by Ungerboeck [1]. Our aim is to find simple sufficient conditions for the resulting spectrum to be equal to the spectrum of an uncoded signal.

The encoder is modeled as in [2]. A finite-state sequential machine with N states is driven by a stationary sequence $(a_n)_{n=-\infty}^{\infty}$ of independent source symbols taking on values $0, 1, \dots, L-1$, and emitted every T seconds. If $(\sigma_n)_{n=-\infty}^{\infty}$ denotes the sequence of states of the sequential machine, and $(u_n)_{n=-\infty}^{\infty}$ the sequence of coded symbols, the behavior of the encoder is described by the equations

$$u_n = h(a_n, \sigma_n) \quad (1.1)$$

and

$$\sigma_{n+1} = g(a_n, \sigma_n) \quad (1.2)$$

Eq. (1.1) describes how the encoded symbol u_n depends on the source symbol a_n and on the actual state σ_n of the encoder. Eq. (1.2) tells in which state the sequential machine is forced to move at time instant $(n+1)T$ when it was in state σ_n at time nT and the source symbol a_n is fed to the encoder.

The coded sequence (u_n) is then sent into ^{n linear} modulator. This can be seen as a mapping of the symbols u_n into waveforms which are output sequentially. If the coded symbols take on values in the set $\{0, 1, \dots, M-1\}$ and $q(t)$ is the waveform available

UNGERBOECK CODES DO NOT SHAPE THE SIGNAL POWER SPECTRUM

by

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ABSTRACT

In this paper we find simple sufficient conditions for the spectrum of Ungerboeck-coded signals to be equal to the spectrum of uncoded signals. It is shown that a carefully designed Ungerboeck code does not shape the spectrum.

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Appendix E

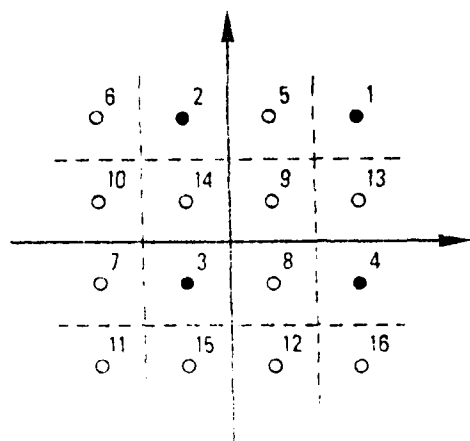


Fig.1 - 16-QAM as GOC

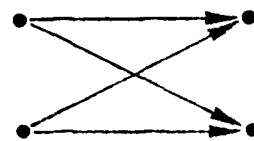


Fig.2 - A 2-state trellis

A	B	C	D
a a a 0	a a 0 a	a 0 a a	0 a a a
0 -a a a	-a a -a 0	a a 0 -a	-a 0 -a a
a 0 -a a	0 -a -a a	-a a a 0	-a a 0 -a
a -a 0 -a	-a 0 a a	0 -a a -a	a a -a 0
-a -a -a 0	-a -a 0 -a	-a 0 -a -a	0 -a -a -a
0 a -a -a	a -a a 0	-a -a 0 a	a 0 a -a
-a 0 a -a	0 a a -a	a -a -a 0	a -a 0 a
-a a 0 a	a 0 -a -a	0 a -a a	-a -a a 0

Fig.3 - Pair partition of a 4D alphabet with M=32

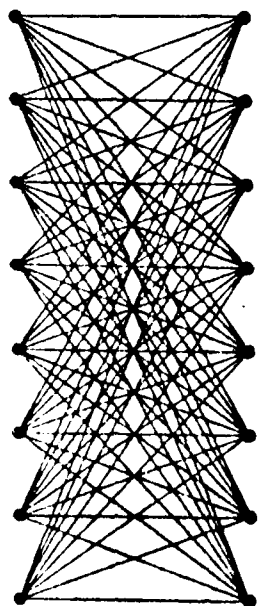


Fig.4 - An 8-state trellis

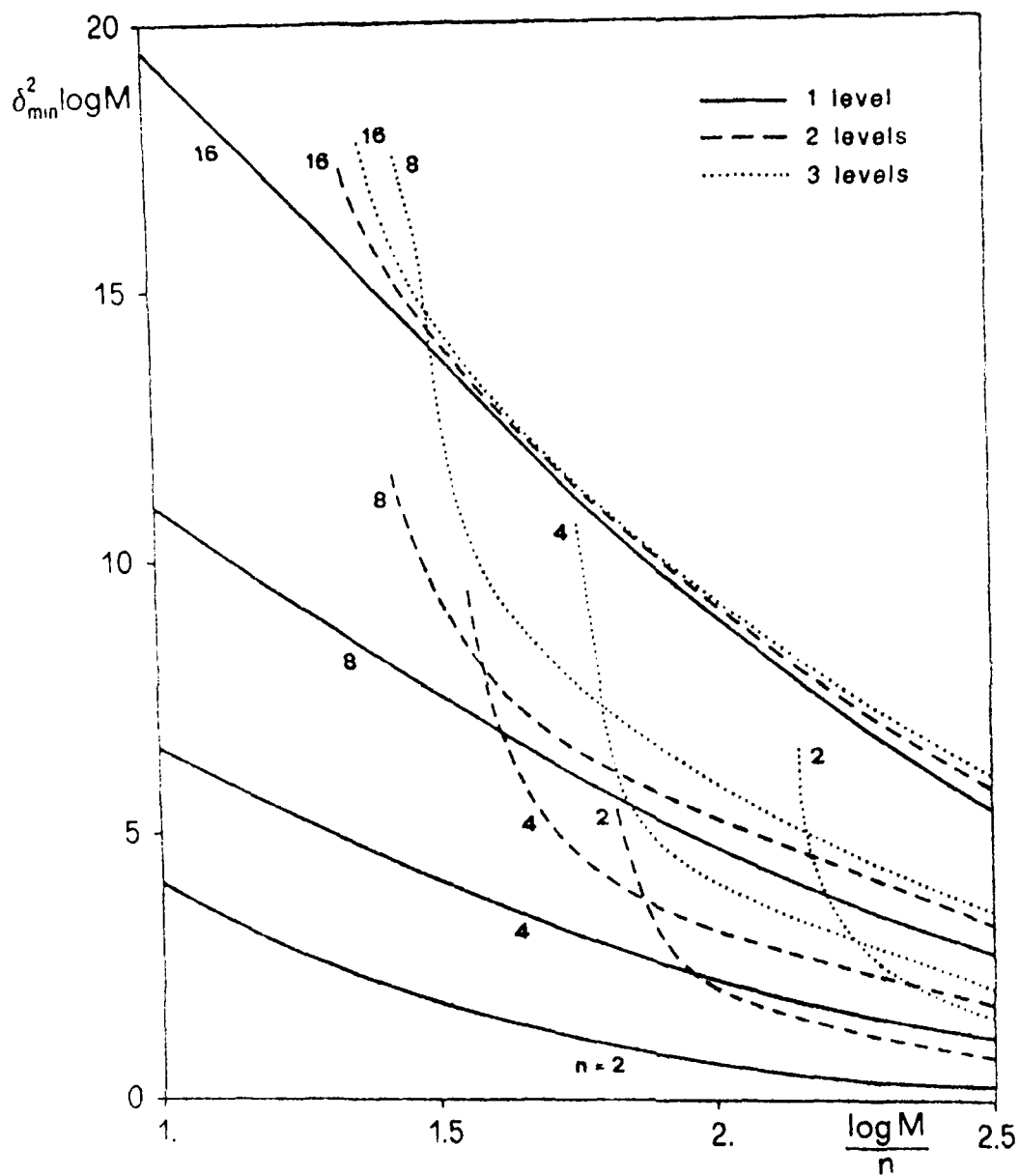


Fig.0 - Bounds on n-dimensional M-ary signal sets

code with two energy levels, obtained from the initial set of 4 vectors:

$$\begin{array}{cccc} c & c & c & 0 \\ -b & c & 0 & c \\ c & -b & 0 & c \\ c & 0 & -b & c \end{array}$$

with $c=0.389$ and $b=0.939$. Applying to the initial set the same group of matrices of the previous example, we get a 128-symbol alphabet. 32 symbols have energy $3c^2$, and 96 symbols have energy $b^2 + 3c^2$. The average energy is

$$[96(b^2 + 3c^2) + 32(3c^2)]/128 = 1.$$

This alphabet can be used to design an Ungerboeck code with rate 6/7. Fig. 4 shows the 8-state trellis employed. A fair partition of the alphabet into 16 subsets of 8 vectors each stems from the subgroup $\{I, -I\}$, where I is the $n \times n$ identity matrix. For example, the first coset includes the initial set and its negative. We get

$$d_{free}^2 = 0.604$$

which is 3.44 dB greater than the minimum distance achieved by using two independent 8-PSK signals with the same energy.

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K different values, and the average error probability is given by

$$P(e) = \frac{1}{K} \sum_{j=1}^K P(e|x_j^{(n)})$$

Fig.1 shows a familiar example of a signal alphabet (the conventional 16-QAM) being a GGC. It has parameters $M=16$ and $n=2$. Points 1,2,3,4 denote the four vectors in the initial set. The matrices generating the code are those associated to plane rotations by multiples of 90° . There are three different energy levels, and three different shapes of ML regions (their boundaries are shown in Fig.1). consider now the set of M vectors in a GGC, and a partition of it into m disjoint subsets X_1, \dots, X_m . For each subset X_i , we can define the "intradistance set" as the set of all the distances among pairs of vectors in X_i . We say that a partition of a GGC is "fair" if all the subsets X_i include the same number of vectors, and their intradistance sets are equal.

A way to generate a fair partition of a GGC is the following. Partition the matrix group G into the cosets of a subgroup H, and apply each coset to the initial set of vectors. It can be shown that the resulting partition of the GGC is fair. For example, by partitioning the rotation group used for the GGC of Fig.1 into the two cosets associated to the rotations 0° , 180° and 90° , -90° , respectively, the code is fairly partitioned into the two subcodes $\{1,2,3,4,9,10,11,12\}$ and $\{5,6,7,8,13,14,15,16\}$.

3.0 MULTIDIMENSIONAL UNGERBOECK CODES

We shall now see how an Ungerboeck code can be designed using a multidimensional alphabet generated as in Section 2. For simplicity, we shall restrict our attention to codes with rate $m/(m+1)$, $m=\log_2 M$. Such codes can be specified by giving an N-state trellis, with M branches stemming from every node and M branches reaching every node. The branches are associated to the elements of an alphabet with 2M elements (see [6] for more details). Although no formal proof has been found yet, for a good code it is conjectured that the assignment of symbols to the branches should exhibit a regular and symmetrical structure. This in turn can be obtained by assigning to each node and to the branches connecting any

pair of nodes a subset of symbols obtained from a fair partition of the alphabet. In fact, all the designs of Ungerboeck codes known to the authors exhibit this kind of symmetry, although no systematic procedure to achieve it has been proposed before

4.0 SOME EXAMPLES

We shall now describe in some details two examples of designs of four-dimensional Ungerboeck codes. No attempt at optimizing the designs has been undertaken yet, so these examples should be taken as an illustration of the concepts presented before and not as a list of good 4D codes.

The first example originates from a group code, obtained by permuting the components, and replacing them with their negatives, in the initial vector $(a,a,a,0)$, $a=1/\sqrt{3}$. This alphabet includes 32 vectors. The 2-state trellis of Fig.2 can be used to design an Ungerboeck code with rate 4/5, and hence transmit 4 bits per alphabet symbol.

Fig.3 shows a fair partition of the alphabet in four subsets of 8 vectors each. This partition is obtained as follows: denote by α the orthogonal matrix whose effect on a vector is to cyclically shift its components to the right by one position, and to change sign to the second component. Then the set

$$H = \{\alpha^0, \alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6, \alpha^7\}$$

is a cyclic subgroup of the group G generating the alphabet, and its cosets generate the fair partition of Fig.3. By associating the first node of the trellis the subcodes A and B, and to the second node the subcodes C and D, the free distance of this code is

$$d_{free}^2 = 4a^2 = 1.33$$

If this figure is compared to the minimum distance achieved by transmitting the same amount of information over the same number of dimensions using two independent 4-PSK signals, we see that an energy saving of 1.24 dB is obtained.

Consider now a generalized group

$\sigma_n \backslash a_n$	S_1	S_2	S_3	S_4
0	w^0	w^1	w^2	w^3
1	w^4	w^5	w^6	w^7
2	w^2	w^3	w^0	w^1
3	w^6	w^7	w^4	w^5

$$w = e^{j\pi/4}$$

(a)

$\sigma_n \backslash a_n$	S_1	S_2	S_3	S_4
0	S_1	S_3	S_1	S_3
1	S_1	S_3	S_1	S_3
2	S_2	S_4	S_2	S_4
3	S_2	S_4	S_2	S_4

(b)

Fig. 1 - Description of a 4-state Ungerboeck code with $L=4, M=8$.

(a) Values of the function $h(\cdot, \cdot)$

(b) Values of the function $g(\cdot, \cdot)$

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